Worcester County Mathematics League

Varsity Meet 3 - February 15, 2023

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 3 - February 15, 2023
Answer Key

Round 1 - Similarity and Pythagorean Theorem

1. 24
2. 4
3. $6\sqrt{5}$

Round 2 - Algebra I

1. 42
2. $x > 2$ or $x \in (2, \infty)$
3. $2\sqrt{A^2 - P}$

Round 3 - Functions

1. $-\dfrac{1}{2}$
2. $(-\infty, 30) \cup (30, \infty)$ or $(30, \infty) \cup (-\infty, 30)$
3. $(-1, -10)$ (exact order)

Round 4 - Combinatorics

1. 120
2. 27
3. 5775

Round 5 - Analytic Geometry

1. $2\sqrt{2}$
2. $2\sqrt{30}$
3. $\dfrac{35}{4}$ or 8.75 or $8\dfrac{3}{4}$

Team Round

1. 8
2. 60
3. 33
4. 56
5. $\left(\dfrac{6}{25}, -\dfrac{24}{25}, -\dfrac{101}{25}\right)$ or $(0.24, -0.96, -4.04)$ or $\left(\dfrac{6}{25}, -\dfrac{24}{25}, -4\dfrac{1}{25}\right)$
   (but triplet must be in exact order)
6. $-n^3$
7. 50
8. 12
9. $\left(-\dfrac{2}{5}, \dfrac{29}{5}\right)$ or $\left(-\dfrac{2}{5}, 5\dfrac{4}{5}\right)$ or $(-0.4, 5.8)$
   (but ordered pair must be in exact order)

Worcester County Mathematics League

Varsity Meet 3 - February 15, 2023

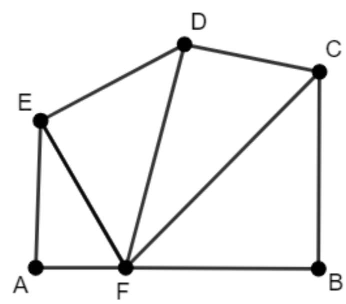
Round 1 - Similarity and Pythagorean Theorem

All answers must be in simplest exact form in the answer section.

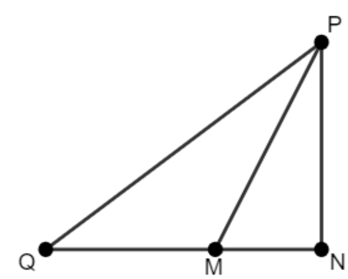
NO CALCULATORS ALLOWED

1. Given that $\Delta CAT \sim \Delta DOG$, $CA = 25$, $CT = 10$, and $DO = 60$; find DG.

2. Given the figure shown to the right with $\overline{EA} \perp \overline{AF}$, $\overline{FE} \perp \overline{ED}$, $\overline{FD} \perp \overline{DC}$, $\overline{FB} \perp \overline{BC}$, $m\angle BFC = m\angle DFE = 45°$, $m\angle FCD = 60°$, and $AE = 3$; find BC.



3. Given the figure at right with $\angle QPM \cong \angle MPN$, $\angle N$ is a right angle, $PQ = 20$, and $PN = 12$; find PM.



<u>ANSWERS</u>

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Bromfield, St. John's, Hudson

Worcester County Mathematics League

Varsity Meet 3 - February 15, 2023

Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. In a group of horses and chickens, the number of legs is 84 more than twice the number of heads. How many horses are there?

2. Solve the following inequality:

$$|2x - 3| > |2x - 5|$$

3. If the average of two numbers is A and the product of two numbers is P, what is the absolute value of the difference of the two numbers? Express your answer in terms of A and P.

ANSWERS

(1 pt) 1.____________________

(2 pts) 2. {____________________}

(3 pts) 3.____________________

Leicester, Algonquin, Leicester/QSC

Worcester County Mathematics League

Varsity Meet 3 - February 15, 2023

Round 3 - Functions

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. If the inverse function of $f(x) = \frac{4x+5}{3}$ is written in the form $f^{-1}(x) = \frac{ax+b}{c}$, find the value of $\frac{a+b}{c}$.

2. Find the range of the function $g(x) = \frac{5x^2-45}{x-3}$. Express your answer as the union of sets using interval notation.

3. Let $f(x) = (x+5)^3 - 5$ and $g(x) = ax + b$. If $g(f(x)) = f(g(x))$, $a \neq 0$, and $a \neq 1$, find the ordered pair (a, b).

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $(a, b) =$ ________________

Doherty, Tahanto, QSC

Worcester County Mathematics League
Varsity Meet 3 - February 15, 2023
Round 4 - Combinatorics

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Six runners (Ava, Bob, Clay, Daisy, Edsel, and Fran) compete for prizes in a race. How many ways are there to award the prizes for first, second, and third place if there are no ties?

2. How many distinct ordered pairs (m, n) can be formed, where m and n are chosen from the set $\{1, 2, 3, 4, 5, 6\}$, $4 < m + n \leq 10$, and m and n may be the same or different numbers?

3. Mr. Badoian will divide the twelve students of his calculus class into teams. How many ways can he divide the class into three teams, each with four students?

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Tantasqua, Worc. Acad., Southbridge/QSC

Worcester County Mathematics League

Varsity Meet 3 - February 15, 2023

Round 5 - Analytic Geometry

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find the radius of the circle described by the following equation:

$$x^2 + y^2 + 10x - 4y + 21 = 0$$

2. The area of the region bounded by the graph of the relation

$$5x^2 + 6y^2 = 60$$

can be expressed as a multiple of π, that is, $z\pi$. Find z in simplest radical form.

3. Find the positive value of x such that the triangle with vertices $(5, 6)$, $(8, 2)$, and $(x, 11)$ has area equal to 15 square units.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Douglass, Quaboag, Westborough

Worcester County Mathematics League
Varsity Meet 3 - February 15, 2023
Team Round

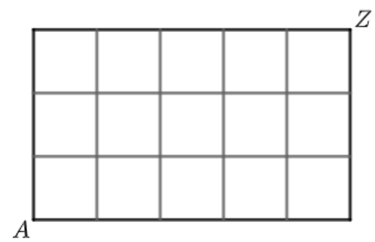
All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. A 25 foot ladder is placed against a vertical wall. The foot of the ladder is 7 feet from the base of the wall along the horizontal ground. If the top of the ladder slips down 4 feet, then how far, in feet, will the foot of the ladder slide (away from the wall)?

2. Train Alpha travels at a speed 15 miles per hour (mph) greater than that of train Beta. If train Alpha travels 150 miles in the same time train Beta travels 120 miles, what is the speed of train Beta (in mph)?

3. Let $f(x) = x^2 + bx + c$, where $f(1) = 9$ and $f(3) - f(2) = 8$. Find $f(4)$.

4. How many ways are there to travel from point A to point Z on the grid shown at right if movement is restricted to moving either one unit up or one unit to the right at each step?

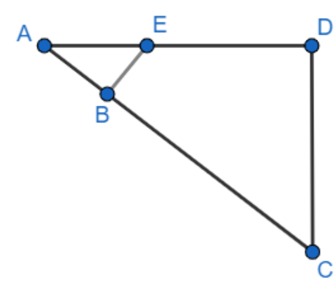


5. A parabola passes through the points $(7, 1)$ and $(-3, 1)$ and has a minimum value of -5. If the equation of the parabola is written in the form $y = Ax^2 + Bx + C$, find the ordered triple (A, B, C).

6. Simplify the product shown below:

$$\left(\frac{n(n-1)}{2} - \frac{n(n+1)}{2}\right)\left(\frac{n(n-1)}{2} + \frac{n(n+1)}{2}\right)$$

7. In the figure at right: $\angle ABE \cong \angle ADC$, $AE = 5$, $ED = 8$, and $CD = 10$. What is the value of $AC \cdot BE$?



8. A number of married couples attends a party, where each person attends with their spouse. Each person shakes the hand of everyone at the party exactly once, except themself and their spouse. If there were a total of 264 handshakes, how many couples attended the party?

9. A line is tangent to a circle at the point $(2, 5)$, where the circle is defined by the equation $x^2 + y^2 = 29$. If the equation of the line is expressed in slope intercept form $(y = mx + b)$, find (m, b).

Worcester County Mathematics League
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Team Round Answer Sheet

ANSWERS

1. ______________________ feet

2. ______________________ miles per hour

3. ______________________

4. ______________________

5. $(A, B, C) =$ ____________

6. ______________________

7. ______________________

8. ______________________

9. $(m, b) =$ ______________

Bromfield, Sheherd Hill, Tantasqua, Tantasqua, Tantasqua, Bancroft, Pomfret, Algonquin, Douglas

Worcester County Mathematics League
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Answer Key

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1. $-\dfrac{1}{2}$
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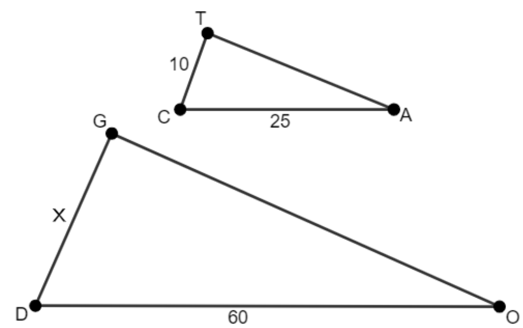
Round 1 - Similarity and Pythagorean Theorem

1. Given that $\Delta CAT \sim \Delta DOG$, $CA = 25$, $CT = 10$, and $DO = 60$; find DG.

Solution:
Draw and label the two similar triangles such that the corresponding sides are in corresponding positions, as shown at right. Next, label the drawing wth the given information, where x is the desired length DG. Now write the proportion of corresponding side lengths:



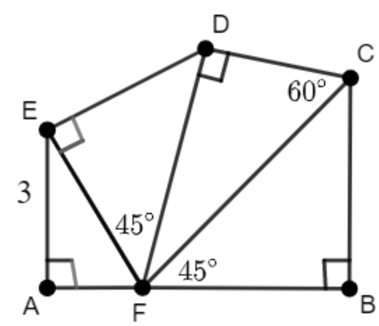
$$\frac{x}{10} = \frac{60}{25} = \frac{12}{5}$$

Finally, multiply the equation by 10 and solve: $x = 10\left(\frac{12}{5}\right) = 12\left(\frac{10}{5}\right) = 12 \cdot 2 = \boxed{24}$.

2. Given the figure shown below with $\overline{EA} \perp \overline{AF}$, $\overline{FE} \perp \overline{ED}$, $\overline{FD} \perp \overline{DC}$, $\overline{FB} \perp \overline{BC}$, $m\angle BFC = m\angle DFE = 45°$, $m\angle FCD = 60°$, and $AE = 3$; find BC.

Solution:

Mark the given information on the figure, as shown at right. Then note that the angle measures in a triangle sum to 180°, so $m\angle DFC + m\angle FCD + m\angle CDF = m\angle DFC + 60° + 90°$, and $m\angle DFC = 30°$. Next, note that measures of angles that form a line sum to 180°, so $m\angle AFE +\ m\angle EFD + m\angle DFC + m\angle CFB = m\angle AFE + 45° + 30° + 45° = 180°$. Therefore $m\angle AFE = 180 - 45 - 45 - 30 = 180 - 90 - 30 = 60°$.



This problem is solved by applying the ratios of the side lengths of 30-60-90 ($1 : \sqrt{3} : 2$) and 45-45-90 ($1 : 1 : \sqrt{2}$) triangles for each of the four triangles (ΔAEF, ΔFED, ΔFDC, and ΔFBC) in succession.

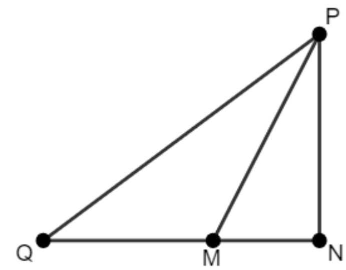
$$\frac{AE}{EF} = \frac{3}{EF} = \frac{\sqrt{3}}{2}, \text{ so } EF = \frac{3 \cdot 2}{\sqrt{3}} = 2\left(\frac{3}{\sqrt{3}}\right) = 2\sqrt{3}$$

$$\frac{DF}{EF} = \frac{DF}{2\sqrt{3}} = \frac{\sqrt{2}}{1}, \text{ so } DF = (2\sqrt{3})\sqrt{2} = 2(\sqrt{3}\sqrt{2}) = 2\sqrt{6}$$

$$\frac{CF}{DF} = \frac{CF}{2\sqrt{6}} = \frac{2}{\sqrt{3}}, \text{ so } CF = 2\frac{2\sqrt{6}}{\sqrt{3}} = 4\sqrt{\frac{6}{3}} = 4\sqrt{2}$$

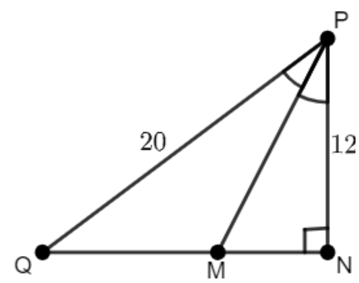
$$\frac{BC}{CF} = \frac{BC}{4\sqrt{2}} = \frac{1}{\sqrt{2}}, \text{ and } BC = \frac{4\sqrt{2}}{\sqrt{2}} = \boxed{4}.$$

3. Given the figure at right with $\angle QPM \cong \angle MPN$, $\angle N$ is a right angle, $PQ = 20$, and $PN = 12$; find PM.



Solution:

First label the diagram with the given information, as shown at right. Next, apply the Pythagorean Theorem to ΔNPQ:



$$(NQ)^2 + (12)^2 = 20^2$$
$$NQ^2 = 400 - 144 = 256$$

and $NQ = \sqrt{256} = 16$. Alternatively, note that the side lengths of ΔNPQ are in proportion $3 : 4 : 5$.

Next, let $MN = y$, so that $QM = NQ - MN = 16 - y$. The angle bisector theorem (the angle bisector of a triangle angle divides the opposite side into segments whose lengths are in proportion to the lengths of the two adjacent sides) can be applied because $\overline{PM}$ is the angle bisector of $\angle NPQ$. Write the proportion:

$$\frac{QM}{MN} = \frac{PQ}{PN}, \text{ or } \frac{16-y}{y} = \frac{20}{12} = \frac{5}{3}$$

Solve for y: $5y = 3(16 - y) = 48 - 3y$; so $8y = 48$, and $MN = y = \frac{48}{8} = 6$.

Finally, find PM by applying the Pythagorean Theorem to ΔMNP:

$$MN^2 + PN^2 = PM^2$$
$$6^2 + 12^2 = 36 + 144 = 180 = PM^2$$

Simplifying, $PM = \sqrt{180} = \sqrt{36 \cdot 5} = \sqrt{36}\sqrt{5} = \boxed{6\sqrt{5}}$.

Round 2 - Algebra I

1. In a group of horses and chickens, the number of legs is 84 more than twice the number of heads. How many horses are there?

Solution: Let h be the number of horses and c be the number of chickens. Horses have four legs and one head. Chickens have two legs and one head. Therefore the number of heads is $h + c$ and the number of legs is $4h + 2c$. Summarize the given information in the following equation:

$$4h + 2c = 2(h + c) + 84 = 2h + 2c + 84$$

Subtract $2h + 2c$ from both sides of the equation, leaving $2h = 84$, and solve: $h = \boxed{42}$.

2. Solve the following inequality:

$$|2x - 3| > |2x - 5|$$

Solution: Consider the two critical points, $x = \frac{3}{2}$ and $x = \frac{5}{2}$, where the two absolute value expressions change from negative to positive. These two points divide the x-axis into three regions: $x < \frac{3}{2}$, $\frac{3}{2} \le x \le \frac{5}{2}$, and $x > \frac{5}{2}$. In the first region, the arguments to both absolute value functions are less than zero; the equivalent relation is therefore $-2x + 3 > -2x + 5$, or equivalently $3 > 5$, which is never true. Therefore there are no values of $x < \frac{3}{2}$ for which the given relation is true. In the third region, the arguments to both absolute value functions are greater than zero. The equivalent relation is $2x - 3 > 2x - 5$, or $-3 > -5$, which is always true. Therefore the statement is true for all values of $x > \frac{5}{2}$.

In the middle region, the expression in the left hand absolute value function is positive, while the argument to the right hand absolute value function is negative. The equivalent relation in this region is $2x - 3 > 5 - 2x$. Add $2x + 3$ to both sides to yield the relation $4x > 8$, or $\boxed{x > 2}$. Note that because the relation is true for all $x > \frac{5}{2}$ and it is true for all $x > 2$ and $x \le \frac{5}{2}$, it is therefore true for all $x > 2$.

3. If the average of two numbers is A and the product of two numbers is P, what is the absolute value of the difference of the two numbers? Express your answer in terms of A and P.

Solution: Let m and n be the two numbers. Then $A = \frac{m+n}{2}$, or $2A = m + n$, and $P = mn$. Squaring $2A$ results in $(2A)^2 = 4A^2 = (m + n)^2 = m^2 + 2mn + n^2$. Therefore $4A^2 - 4P = m^2 + 2mn + n^2 - 4mn = m^2 - 2mn + n^2 = (m - n)^2$. Finally, $|m - n| = \sqrt{(m - n)^2} = \sqrt{4A^2 - 4P} = \sqrt{4(A^2 - P)} = \sqrt{4}\sqrt{A^2 - P} = \boxed{2\sqrt{A^2 - P}}$.

Round 3 - Functions

1. If the inverse function of $f(x) = \frac{4x+5}{3}$ is written in the form $f^{-1}(x) = \frac{ax+b}{c}$, find the value of $\frac{a+b}{c}$.

Solution: Note that $f(f^{-1}(x)) = x$ by the definition of an inverse function. Then $x = \frac{4f^{-1}(x)+5}{3}$. Multiply both sides of this equation by 3 and subtract 5, resulting in $4f^{-1}(x) = 3x - 5$. Then $f^{-1}(x) = \frac{3x-5}{4}$ and $a = 3$, $b = -5$, $c = 4$. The requested expression $\frac{a+b}{c} = \frac{3-5}{4} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$.

2. Find the range of the function $g(x) = \frac{5x^2-45}{x-3}$. Express your answer as the union of sets using interval notation.

Solution: First note that $g(x)$ is a ratio of two polynomials and that the numerator polynomial can be factored:

$$g(x) = \frac{5(x^2-9)}{x-3} = \frac{5(x+3)(x-3)}{x-3} = 5(x+3)$$

where the last step cancelled the common factor $x - 3$ from the numerator and denominator Thus, $g(x) = 5(x + 3)$ for all values of x in the domain and $g(x)$ is a linear function. Returning to the original expression, note that the domain of $g(x)$ must exclude the value $x = 3$ to prevent dividing by zero. Note that a linear function is 1-1 and onto, so that the only value missing from the range is $g(3) = 5(3 + 3) = 5 \cdot 6 = 30$ Therefore the range of g is all Real values not equal to 30. In other words, it is all values less than 30 or greater than 30. Expressed in interval notation, the range is equal to $\boxed{(-\infty, 30) \cup (30, \infty)}$.

3. Let $f(x) = (x+5)^3 - 5$ and $g(x) = ax + b$. If $g(f(x)) = f(g(x))$, $a \neq 0$, and $a \neq 1$, find the ordered pair (a, b).

Solution: First apply the given definitions of $f(x)$ and $g(x)$: $g(f(x)) = a\left((x+5)^3 - 5\right) + b$ and $f(g(x)) = (ax+b+5)^3 - 5$. Recall that a cubic binomal expands to $(m+n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$. Therefore:

$$\begin{aligned}
g(f(x)) &= a(x^3 + 3(5)x^2 + 3(5^2)x + 5^3 - 5) + b \\
&= ax^3 + 15ax^2 + 75ax + a(125 - 5) + b \\
&= ax^3 + 15ax^2 + 75ax + 120a + b \\
f(g(x)) &= (ax)^3 + 3(b+5)(ax)^2 + 3(b+5)^2(ax) + (b+5)^3 - 5 \\
&= a^3x^3 + 3(b+5)a^2x^2 + 3(b^2 + 10b + 25)ax + b^3 + 3(5)b^2 + 3(5^2)b + 5^3 - 5 \\
&= a^3x^3 + 3a^2(b+5)x^2 + 3a(b^2 + 10b + 25)x + b^3 + 15b^2 + 75b + 120
\end{aligned}$$

and both functions are degree 3 polynomials in x. In order for two polynomials to be equal, each of their respective coefficients must be equal. Equating the x^3 coefficients results in $a^3 = a$. Collect terms on one side and factor: $a^3 - a = a(a^2 - 1) = a(a-1)(a+1) = 0$. Thus, $a = 0, 1,$ or -1. The first two values were excluded in the problem statement, so $a = -1$.

Equating the x^2 coefficients and setting $a = -1$ results in $15(-1) = 3(-1)^2(b+5) = 3b + 15$, or $3b = -15 - 15 = -30$. Finally, $b = \frac{-30}{3} = -10$, and $(a, b) = \boxed{(-1, -10)}$.

It remains to check that the final two coefficients are equal, or to check that $g(f(x)) = f(g(x))$ with $(a, b) = (-1, -10)$. Using the second method,

$$\begin{aligned}
f(g(x)) &= (-x - 10 + 5)^3 - 5 = (-x - 5)^3 - 5 = (-(x+5))^3 - 5 = -(x+5)^3 - 5 \\
g(f(x)) &= -((x+5)^3 - 5) - 10 = -(x+5)^3 + 5 - 10 = -(x+5)^3 - 5
\end{aligned}$$

so $g(f(x)) = f(g(x))$ for the boxed values of (a, b) and all x, completing the verification.

Round 4 - Combinatorics

1. Six runners (Ava, Bob, Clay, Daisy, Edsel, and Fran) compete for prizes in a race. How many ways are there to award the prizes for first, second, and third place if there are no ties?

Solution: This problem asks for the number of ways to choose three elements chosen from a set of six elements in a specific order. More formally, this is the number of permutations of three elements chosen from six distinct elements, or ${}_6P_3 = \frac{6!}{(6-3)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = \boxed{120}$.

The same answer can be found by counting the ways to award the prizes in order. That is, there are six possible first prize winners. Once the first prize winner is chosen, there five possible second prize winners. Once the first and second prizes are chosen, there are four possible third prize winners, so the total number of ways to award the prizes is $6 \cdot 5 \cdot 4$.

2. How many distinct ordered pairs (m, n) can be formed, where m and n are chosen from the set $\{1, 2, 3, 4, 5, 6\}$, $4 < m + n \leq 10$, and m and n may be the same or different numbers?

Solution: It is easier to count the number of excluded ordered pairs and subtract that number from the total possible number of ordered pairs than to directly count the number of ordered pairs that fit the restriction.

Proceeding with this method, note that there are six choices for m and six choices for n. That makes $6 \cdot 6 = 36$ possible distinct ordered pairs (m, n) when m and n are unrestricted. Next, note that the six ordered pairs $(1, 1), (1, 2), (2, 1), (3, 1), (1, 3)$ and $(2, 2)$ need to be excluded from the count because $m + n \leq 4$ for each of those pairs. Also, note that the three ordered pairs $(5, 6), (6, 5),$ and $(6, 6)$ need to be excluded from the count because $m + n > 10$. Therefore the number of ordered pairs with the restriction $4 < m + n \leq 10$ is equal to $36 - 6 - 3 = \boxed{27}$.

3. Mr. Badoian will divide the twelve students of his calculus class into teams. How many ways can he divide the class into three teams, each with four students?

Solution: There are multiple ways to arrive at the correct answer. One way is to consider that the number of ways to order the 12 students is 12! and then assign the first four students in the list to one team, the second four students to another team, and the last four students to the third team. Because the order of the students within a team doesn't matter, this count needs to be divided by 4! three times, once for each of the three teams. That is, there are 4! ways to order the four students in a team. Finally, there is no distinction between the three teams, so the count needs to be divided by 3! because the three teams can be ordered 3! ways.

In summary, the total number of ways to divide the 12 students into three teams is

$$\frac{12!}{(4!)^3 \cdot 3!}.$$

Expand this expression and cancel terms common to the numerator and denominator to find the integer answer:

$$\begin{aligned}
\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} &= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 2} \\
&= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{12 \cdot 8 \cdot 6 \cdot 3 \cdot 2} \\
&= \frac{11 \cdot (5 \cdot 2)(3 \cdot 3) \cdot 7 \cdot 5}{3 \cdot 2} \\
&= 11 \cdot 5 \cdot 3 \cdot 7 \cdot 5 \\
&= 11 \cdot 21 \cdot 25 = 231 \cdot 25 = \boxed{5775}
\end{aligned}$$

Round 5 - Analytic Geometry

1. Find the radius of the circle described by the following equation:

$$x^2 + y^2 + 10x - 4y + 21 = 0$$

Solution: Both the radius r and the center (h, k) of the circle can be easily found by transforming the given equation into the standard form for the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

Proceed by subracting 21 from both sides of the given equation and completing the square for the two quadratic functions of x and y:

$$x^2 + 10x + \left(\frac{10}{2}\right)^2 + y^2 - 4y + \left(\frac{4}{2}\right)^2 = -21 + 5^2 + 2^2$$
$$(x + 5)^2 + (y - 2)^2 = -21 + 25 + 4 = 8 = r^2$$

Now $r^2 = 8$, so $r = \sqrt{8} = \sqrt{4} \cdot \sqrt{2} = \boxed{2\sqrt{2}}$.

2. The area of the region bounded by the graph of the relation

$$5x^2 + 6y^2 = 60$$

can be expressed as a multiple of π, that is, $z\pi$. Find z in simplest radical form.

Solution: First, divide both sides of the original equation by 60:

$$\frac{x^2}{12} + \frac{y^2}{10} = 1.$$

This equation is the standard form for an ellipse centered at $(0, 0)$. One axis of the the ellipse, of length $2r_1$, lies on the x-axis and the other axis, of length $2r_2$, lies on the y-axis:

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1.$$

Thus, $r_1 = \sqrt{12} = 2\sqrt{3}$ and $r_2 = \sqrt{10}$.

The area contained by an ellipse is equal to $\pi r_1 r_2$, where the two axes of the ellipse have lengths $2r_1$ and $2r_2$. In this case, the area is equal to $\pi 2\sqrt{3}\sqrt{10} = \pi(2\sqrt{30}) = \pi x$ and $x = \boxed{2\sqrt{30}}$.

3. Find the positive value of x such that the triangle with vertices $(5, 6)$, $(8, 2)$, and $(x, 11)$ has area equal to 15 square units.

Solution: One method to solve this equation is to find an expression for the distance from the vertex $(x, 11)$ to the line containing the other two vertices $(5, 6)$ and $(8, 2)$. This distance is the length of the altitude, that is, the height h of the triangle. Then the length of the base b is the distance between the other two vertices, or $b = \sqrt{(5-8)^2 + (6-2)^2} = \sqrt{3^2 + 4^2} = 5$ (recognizing the $(3, 4, 5)$ Pythagorean triple). The area of the triangle is $15 = \frac{1}{2}bh = 15 = \frac{1}{2}5h$, so $h = \frac{15 \cdot 2}{5} = 6$.

Now the distance d between a point (h, k) and a line written in the form $Ax + By + C = 0$ is given by the following equation:

$$d = \frac{Ah + Bk + C}{\sqrt{A^2 + B^2}}$$

In this case, $d = h = 6$, $(h, k) = (x, 11)$ and the line has slope $\frac{6-2}{5-8} = -\frac{4}{3}$, using the equation for the slope between $(5, 6)$ and $(8, 2)$. Therefore the line is defined by the equation $y = -\frac{4}{3}x + b$. Plug in $(5, 6)$ to find b: $6 = -\frac{4}{3}5 + b = -\frac{20}{3} + b$. Then $b = 6 + \frac{20}{3} = \frac{38}{3}$, and $y = -\frac{4}{3}x + \frac{38}{3}$. To convert this equation to the desired form, add $\frac{4}{3}x - \frac{38}{3}$ to both sides of the equation and multiple by 3: $4x + 3y - 38 = 0$.

We found that $A = 4$, $B = 3$, and $C = -38$. Plug these into the equation along with $(h, k) = (x, 11)$:

$$d = 6 = \frac{4x + 3 \cdot 11 - 38}{\sqrt{4^2 + 3^2}} = \frac{4x - 5}{5}$$

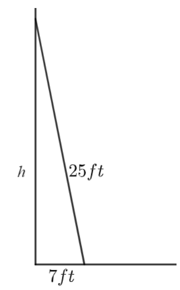
Finally, multiply both sides by 5, add 5, and divide by 4 to find $x = \frac{6 \cdot 5 + 5}{4} = \boxed{\frac{35}{4}}$.

Team Round

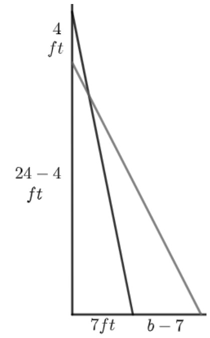
1. A 25 foot ladder is placed against a vertical wall. The foot of the ladder is 7 feet from the base of the wall along the horizontal ground If the top of the ladder slips down 4 feet, then how far, in feet, will the foot of the ladder slide (away from the wall)?

Solution:

Start by drawing a picture, shown at right. The wall and the ground form two legs of a right triangle, where the ladder is the hypotenuse. If h is the height of the ladder above the ground, that is, the distance from the base of the wall to the point where the ladder rests against the wall, then $h^2 + 7^2 = 25^2$ by the Pythagorean Theorem. Then $h^2 = 25^2 - 7^2 = 625 - 49 = 576 = 24^2$, so $h = 24$. (Note that the lengths of the three sides are a Pythagorean triple $(7, 24, 25)$).



Redraw the picture as shown at right. The ladder has slipped 4 feet down the wall, so that the height of the ladder is now $24 - 4 = 20$ feet from the ground. The ladder is still 25 feet long, of course. Now the ladder is distance b from the base of the wall, and $b^2 + 20^2 = 25^2$. Solve for b: $b^2 = 25^2 - 20^2 = 625 - 400 = 225 = 15^2$, so $b = 15$. (Again, the sides of the triangle are a Pythagorean triple $(15, 20, 25)$).



Finally, the distance that the ladder slides away from the wall is $b - 7 = 15 - 7 = \boxed{8}$ feet.

2. Train Alpha travels at a speed 15 miles per hour (mph) greater than that of train Beta. If train Alpha travels 150 miles in the same time train Beta travels 120 miles, what is the speed of train Beta (in mph)?

Solution: Let x be the speed of train Beta in miles per hour. Then the speed of train Alpha is $x + 15$. The time of travel is equal to the distance traveled divided by the speed. Set the ratio of distance to speed for Train Alpha equal to the distance to speed ratio for Train Beta:

$$\frac{150}{x + 15} = \frac{120}{x}$$

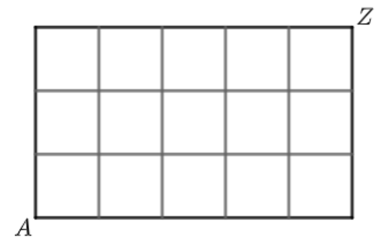
Note that the units of the left and right hand sides of the equation match because both are ratios of miles to miles per hour, and both ratios have units of hours traveled.

Now, multiply both sides of the equation by $x(x+15)$, which results in $150x = 120(x+15)$. Solving for x: $150x = 120x + 120 \cdot 15$, $150x - 120x = 30x = 120 \cdot 15$, and $x = \frac{120}{30} \cdot 15 = 4 \cdot 15 = \boxed{60}$ mph.

3. Let $f(x) = x^2 + bx + c$, where $f(1) = 9$ and $f(3) - f(2) = 8$. Find $f(4)$.

Solution: First, replace x by 1 in $f(x)$, so $f(1) = 9 = 1^2 + b \cdot 1 + c = 1 + b + c$, and $b + c = 9 - 1 = 8$. In a similar manner, $f(2) = 2^2 + 2b + c = 4 + 2b + c$, and $f(3) = 9 + 3b + c$. Then $f(3) - f(2) = 8 = 9 + 3b + c - (4 + 2b + c) = 9 - 4 + 3b - 2b + c - c = 5 + b$. Solving, $5 + b = 8$, so $b = 8 - 5 = 3$, and $b + c = 3 + c = 8$, so $c = 8 - 3 = 5$. Finally, $f(4) = 4^2 + 4b + c = 16 + 4b + c = 16 + 4 \cdot 3 + 5 = 16 + 12 + 5 = \boxed{33}$.

4. How many ways are there to travel from point A to point Z on the grid shown at right if movement is restricted to moving either one unit up or one unit to the right at each step?



Solution: The path from A to Z must include 3 vertical steps up and 5 horizontal steps to the right, for a total of 8 steps. The 3 vertical steps may be taken at any stage, as long as exactly 3 are taken in total. For instance, the first, fourth, and seventh steps might be vertical steps, and this choice (1, 4, 7) forms one distinct path since all other steps must be horizontal. Other choices for vertical steps might be (1, 2, 3), or (6, 7, 8), or (3, 5, 7). Thus, the number of possible paths is equal to the number of ways to choose which 3 of the 8 steps are vertical, or $\binom{8}{3}$ paths.

Calculating:

$$\binom{8}{3} = \frac{8!}{(8-3)!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{6} = 8 \cdot 7 = \boxed{56}$$

Note that counting the number of ways to choose which 5 of the 8 steps are horizontal will also result in 56, since that count is $\binom{8}{5}$, and $\binom{8}{5} = \binom{8}{8-5} = \binom{8}{3}$

5. A parabola passes through the points $(7, 1)$ and $(-3, 1)$ and has a minimum value of -5. If the equation of the parabola is written in the form $y = Ax^2 + Bx + C$, find the ordered triple (A, B, C).

Solution: The parabola has a minimum value of -5 and has two points with identical y-coordinates of 1, so it must open up. It has a vertical axis of symmetry $(x = h)$ that must lie halfway between the two given points $(7, 1)$ and $(-3, 1)$. Therefore h is the average of the two x-coordinates and $h = \frac{7-3}{2} = \frac{4}{2} = 2$. The minimum value is the y-coordinate of the vertix, so the vertex is $(2, -5)$.

Now we know three points that lie on the parabola. Plug the (x, y) coordinates for these three points into the equation of the parabola $(y = Ax^2 + Bx + C)$ to create a system of three linear equations that can be used to solve for (A, B, C):

$$\begin{cases} 2^2A + 2B + C = -5 \\ (-3)^2A - 3B + C = 1 \\ 7^2A + 7B + C = 1 \end{cases} \quad \text{or} \quad \begin{cases} 4A + 2B + C = -5 \\ 9A - 3B + C = 1 \\ 49A + 7B + C = 1 \end{cases}$$

These equations can be solved by various combinations of elimination and substitution operations. The equivalent operations can be performed on the rows of the augmented matrix that represents this system. The following method transforms the system to a triangular system and then solves for A, B, and C by substitution. The steps shown below are, in order:

- Subtract the first equation from the second and third equations to eliminate C
- Add the second equation to the third equation to eliminate B

$$\begin{bmatrix} 4 & 2 & 1 & -5 \\ 9 & -3 & 1 & 1 \\ 49 & 7 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 1 & -5 \\ 5 & -5 & 0 & 6 \\ 45 & 5 & 0 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 1 & -5 \\ 5 & -5 & 0 & 6 \\ 50 & 0 & 0 & 12 \end{bmatrix}$$

The last matrix represents a system of equations that is equivalent to the original system and is easily solved by substitution:

$$\begin{cases} 4A + 2B + C = -5 \\ 5A - 5B = 6 \\ 50A = 12 \end{cases}$$

Solving the third equation, $A = \frac{12}{50} = \frac{6}{25}$.

Next, substitute $A = \frac{6}{25}$ into the second equation and solve for B:

$$5 \cdot \frac{6}{25} - 5B = 6$$
$$5B = 5 \cdot \frac{6}{25} - 6$$
$$B = \frac{6}{25} - \frac{6}{5} = \frac{6}{25} - \frac{30}{25} = -\frac{24}{25}.$$

Finally, substute the values for A and B into the first equation and solve for C:

$$\begin{aligned} 4 \cdot \frac{6}{25} - 2 \cdot \frac{24}{25} + C = -5 \\ C &= -5 - 4 \cdot \frac{6}{25} + 2 \cdot \frac{24}{25} \\ &= \frac{-125}{25} - \frac{24}{25} + \frac{48}{25} = \frac{-101}{25}. \end{aligned}$$

Thus, $(A, B, C) = \boxed{\left(\frac{6}{25}, -\frac{24}{25}, -\frac{101}{25}\right)}$.

6. Simplify the product shown below:

$$\left(\frac{n(n-1)}{2} - \frac{n(n+1)}{2}\right)\left(\frac{n(n-1)}{2} + \frac{n(n+1)}{2}\right)$$

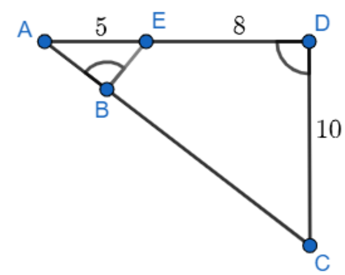
Solution: One way to solve this problem is to note that it is in the factored form of a difference of squares. However, it is easier to first simply add and simplify the rational expressions inside each parenthesis pair. Thus,

$$\begin{aligned} \frac{n(n-1)}{2} - \frac{n(n+1)}{2} &= \frac{n^2 - n - (n^2 + n)}{2} = \frac{n^2 - n^2 - n - n}{2} = \frac{-2n}{2} = -n \\ \frac{n(n-1)}{2} + \frac{n(n+1)}{2} &= \frac{n^2 - n + (n^2 + n)}{2} = \frac{n^2 + n^2 - n + n}{2} = \frac{2n^2}{2} = n^2 \end{aligned}$$

Then the product is $(-n)n^2 = \boxed{-n^3}$.

7. In the figure at right: $\angle ABE \cong \angle ADC$, $AE = 5$, $ED = 8$, and $CD = 10$. What is the value of $AC \cdot BE$?

Solution:
First, label the figure with the given information, as shown at right. It was given that $\angle ABE \cong \angle ADC$. It is also true that $\angle EAB \cong \angle CAD$ because $\angle EAB$ and $\angle CAD$ are the same angle and every angle is congruent to itself. Therefore we have two pairs of congruent angles, and $\Delta ABE \sim \Delta ADC$ by the AA similarity postulate. Corresponding sides of similar triangles are in proportion. We can therefore write a proportion that includes AC and BE:



$$\frac{AC}{AE} = \frac{DC}{BE}$$
$$\frac{AC}{5} = \frac{10}{BE}$$

Crossmultiply the proportion and $AC \cdot BE = 10 \cdot 5 = \boxed{50}$.

8. A number of married couples attends a party, where each person attends with their spouse. Each person shakes the hand of everyone at the party exactly once, except themself and their spouse. If there were a total of 264 handshakes, how many couples attended the party?

Solution: Let n be the number of couples. Then $2n$ people attended the party. Each person shook the hand of $2n - 2$ people. Multiplying the number of people by the number of hands that they shook ($2n(2n-2)$) overcounts the number of handshakes; each handshake is counted twice, once for each person shaking hands. Therefore the product of the number of attendees and the number of hands that each attendee shakes must be divided by 2: $\frac{2n(2n-2)}{2} = n(2n-2) = 2n(n-1)$. Set this expression equal to 264, simplify and factor:

$$2n(n-1) = 264$$
$$n(n-1) = 132$$
$$n^2 - n - 132 = 0$$
$$(n-12)(n+11) = 0$$

and $n = \boxed{12}$. The extraneous solution of -11 is discarded.

9. A line is tangent to a circle at the point $(2, 5)$, where the circle is defined by the equation $x^2 + y^2 = 29$. If the equation of the line is expressed in slope intercept form ($y = mx + b$), find (m, b).

Solution: The circle is centered at the origin. That means that the radius drawn to the point of tangency $(2, 5)$ has a second point $(0, 0)$, and its slope $m_r = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-0}{2-0} = \frac{5}{2}$ using the slope formula. Now the radius drawn to the point of tangency is perpendicular to the tangent, and the product of the slopes of two perpendicular lines is -1. Therefore the slope of the tangent line (m) satisfies $m \cdot \frac{5}{2} = -1$, and $m = -\frac{2}{5}$.

The y-intercept (b) is found by plugging in the point $(x, y) = (2, 5)$ into the slope intercept equation: $5 = 2(-\frac{2}{5}) + b$, $b = 5 + \frac{4}{5} = \frac{25+4}{5} = \frac{29}{5}$. The equation of the line is $y = -\frac{2}{5}x + \frac{29}{5}$, so $(m, b) = \boxed{\left(-\frac{2}{5}, \frac{29}{5}\right)}$.