Worcester County Mathematics League

Varsity Meet 2 - November 28, 2018

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 2 - November 28, 2018
Answer Key

Round 1 - Fractions, Decimals, and Percents

1. 25% or 25 percent
2. $\frac{11}{5}$ or $2\frac{1}{5}$ or 2.2
3. 8 students

Round 2 - Algebra I

1. $t = 6$
2. 9 minutes
3. 13 and 26 (either order)

Round 3 - Parallel Lines and Polygons

1. $x = 12$ and $y = 13$
2. 30° or 30 degrees
3. $CD = \frac{168}{5}$ or $CD = 33\frac{3}{5}$ or $CD = 33.6$

Round 4 - Sequences and Series

1. 8
2. $\frac{2047}{4}$ or $511\frac{3}{4}$ or 511.75
3. $a = 7$ or $a = -1$ (both required)

Round 5 - Matrices and Systems of Equations

1. $\begin{bmatrix} 3 & 5 \\ 1 & 5 \end{bmatrix}$
2. $k = -1$
3. $(-3, 4), \left(3, 4 + 2\sqrt{6}\right), \left(3, 4 - 2\sqrt{6}\right)$ (all required)

Team Round

1. $\frac{409}{990}$
2. $\frac{170}{53}$ or $3\frac{11}{53}$
3. $\frac{5}{54}$
4. 14 km/h = rate in still water
   6 km/h = rate of current
5. 82 or 82°
6. $\frac{531}{380}$ or $1\frac{151}{380}$
7. $x = -3$ or 5, or $x = \{-3, 5\}$
   (NB: "-3 and 5" is not correct)
8. -1
9. $769,915$ (comma optional)

Worcester County Mathematics League
Varsity Meet 2 - November 28, 2018
Round 1 - Fractions, Decimals, and Percents

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If an item at Macy's is on sale for 20% off, what percent would that item's cost need to be increased by in order to bring it back to its original price?

2. Simplify the expression below.

$$\frac{0.\overline{6}\left(\frac{1}{2}+0.6\right)}{-4\left(\frac{3}{4}-0.8\overline{3}\right)}$$

3. At a certain high school, there are x students in the ski club. Half of the ski club is in the chess club, one-third of the chess club is in the Model UN club, and three-fifths of the Model UN club are math team members. If there are fewer than 160 students enrolled in the ski club, everyone has a partner to play against during chess club practice, and every club has a whole number of kids in it, determine the largest possible difference between the number of students in Model UN and the math team.

ANSWERS

(1 pt) 1. ______________ percent

(2 pts) 2. ______________

(3 pts) 3. ______________ students

Quaboag, Grafton, St. Peter-Marian/QSC

Worcester County Mathematics League
Varsity Meet 2 - November 28, 2018
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Solve for t.

$$\frac{1}{5}(2t+3)+\frac{3}{5}(8-3t)=-3$$

2. A zebra leaves a watering hole running at 60 km/h. After the zebra has been running for 90 seconds, a wild dog leaves the watering hole and chases after the zebra at 70 km/h. If both animals maintain their speed, how many minutes does it take the dog to catch up with the zebra after the dog started running?

3. The sum of 22^2 and 19^2 equals the sum of another pair of two-digit numbers squared. Find the two numbers.

ANSWERS

(1 pt) 1. $t =$ ______________

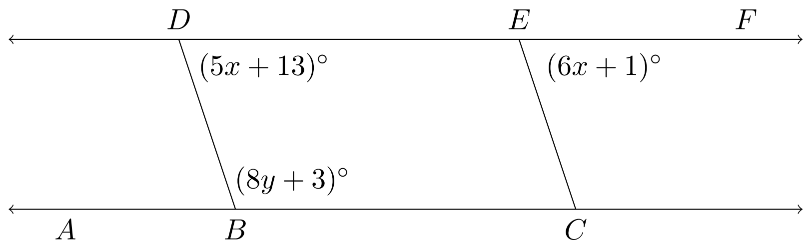
(2 pts) 2. ______________ minutes

(3 pts) 3. __________ and __________

Southbridge, Tantasqua, Burncoat

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. In the diagram below, $\overline{BD} \parallel \overline{CE}$. Find the values of x and y such that $\overline{BC} \parallel \overline{DE}$.



2. If ABCDEF is a regular hexagon, what acute angle is formed by the intersection of $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$? Express your answer in degrees.

3. Given:

   $\overline{BE} \parallel \overline{CD}$

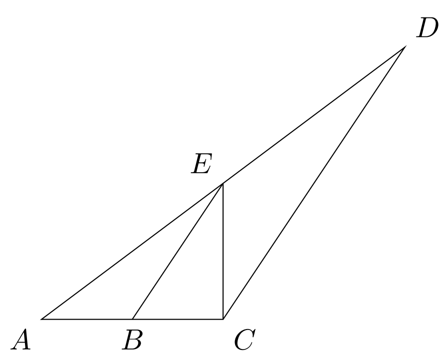
   $\overline{AC} \perp \overline{EC}$

   $AB = 10$

   $AE = 24$

   $BE = 16$

   Find CD.



ANSWERS

(1 pt) 1. $x =$ __________ and $y =$ __________

(2 pts) 2. ____________________ degrees

(3 pts) 3. $CD =$ ____________________

NDA, St. John's, Quaboag

Worcester County Mathematics League
Varsity Meet 2 - November 28, 2018
Round 4 - Sequences and Series

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the first term of the arithmetic sequence if the 5th term is 24 and the 9th term is 40.

2. Evaluate.

$$\sum_{k=1}^{11} 2^{k-3}$$

3. Solve for a.

$$\sum_{x=a}^{40}(2x-5) = 1428$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $a =$ ____________________

Tantasqua, Bromfield, Auburn

Worcester County Mathematics League
Varsity Meet 2 - November 28, 2018
Round 5 - Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find $A(B+C)$ if $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

2. The expression below is equivalent to the identity matrix. Find k.

$$\begin{bmatrix} 2 & 1 & k \\ 3 & k & 4 \end{bmatrix} \begin{bmatrix} k & 4 \\ -2 & -1 \\ -3 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -11 \\ 13 & 4 \end{bmatrix}$$

3. Circle C is centered at $O(2,4)$. Parabola P has a vertex of $(-3,4)$ and an x-intercept of 1. Find all three points of intersection of circle C and parabola P.

ANSWERS

(1 pt) 1. $\begin{bmatrix} \_\_\_\_\_ & \_\_\_\_\_ \\ \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}$

(2 pts) 2. $k =$ ____________________

(3 pts) 3. __________, __________, __________

Tahanto, Quaboag, QSC

Worcester County Mathematics League

Varsity Meet 2 - November 28, 2018

Team Round

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

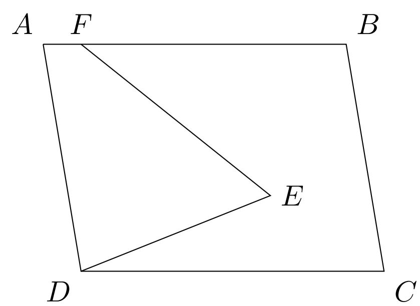
1. Write the following as the ratio of two relatively prime integers: 0.4131313...

2. Evaluate $3+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{2}}}}$.

3. John, Paul, George, Ringo, and Stuart arrive in Hamburg, Germany which has only six hotels. If each person picks a random hotel to stay at, what is the probability that no two people from this group will go to the same hotel?

4. A crew can row a 2 kilometer course downriver (with the current) in just 6 minutes, but it takes the crew 15 minutes to row the same distance upriver (against the current). Find the rate of the crew in still water and the rate of the current, express those rates in kilometers per hour.

5. Given parallelogram ABCD, $\measuredangle E = 62°$, $\measuredangle BFE = 32°$, and $\measuredangle C = 68°$, find $\measuredangle ADE$.



6. Evaluate $\displaystyle\sum_{k=1}^{18}\left(\frac{1}{k}-\frac{1}{k+2}\right)$.

7. Solve for x.

$$\begin{vmatrix} x+4 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & x-2 & 3 \end{vmatrix} = 0$$

8. In the expansion of $(x^2-2)^{15}$, determine the sum of the coefficients.

9. Determine the value of $384,958^2 - 384,957^2$.

Worcester County Mathematics League
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Team Round Answer Sheet

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

4. _______ km/h = rate in still water, _______ km/h = rate of current

5. ______________________________ degrees

6. ______________________________

7. $x =$ ______________________________

8. ______________________________

9. ______________________________

Bromfield, Tahanto, Leicester, Millbury, Auburn, Worcester Academy, Leicester, Worcester Academy, Quaboag

Worcester County Mathematics League
Varsity Meet 2 - November 28, 2018
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2. 9 minutes
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Round 1 - Fractions, Decimals, and Percents

1. If an item at Macy's is on sale for 20% off, what percent would that item's cost need to be increased by in order to bring it back to its original price?

Solution: If P represents the original price and S represents the sale price, then the relationship between P and S is

$$S = .8P = \frac{4}{5}P.$$

From this relationship, it can be shown that

$$P = \frac{5}{4}S = 1.25S,$$

which indicates the original price is a $\boxed{25\%}$ increase from the sale price.

2. Simplify the expression below.

$$\frac{0.\overline{6}\left(\frac{1}{2}+0.6\right)}{-4\left(\frac{3}{4}-0.8\overline{3}\right)}$$

Solution: First, express each decimal and repeating decimal as a fraction. $0.\overline{6}$ is straightforward with $0.\overline{6} = \frac{6}{9} = \frac{2}{3}$, and we know $0.6 = \frac{6}{10} = \frac{3}{5}$, but $0.8\overline{3}$ takes another step. Let $N = 0.8\overline{3}$:

$$\begin{aligned} 10N &= 8.33333\ldots \\ N &= 0.83333\ldots \\ 10N - N = 9N &= 7.5 \\ N &= \frac{7.5}{9} = \frac{75}{90} = \frac{5}{6} \end{aligned}$$

Rewriting the complex fraction and simplifying:

$$\frac{0.\overline{6}\left(\frac{1}{2}+0.6\right)}{-4\left(\frac{3}{4}-0.8\overline{3}\right)} = \frac{\frac{2}{3}\left(\frac{1}{2}+\frac{3}{5}\right)}{-4\left(\frac{3}{4}-\frac{5}{6}\right)} = \frac{\frac{1}{3}+\frac{2}{5}}{-3+\frac{20}{6}} = \frac{\frac{11}{15}}{\frac{2}{6}} = \frac{66}{30} = \boxed{\frac{11}{5} = 2\frac{1}{5} = 2.2}$$

3. At a certain high school, there are x students in the ski club. Half of the ski club is in the chess club, one-third of the chess club is in the Model UN club, and three-fifths of the Model UN club are math team members. If there are fewer than 160 students enrolled in the ski club, everyone has a partner to play against during chess club practice, and every club has a whole number of kids in it, determine the largest possible difference between the number of students in Model UN and the math team.

Solution: If there are x students in the ski club, then the following table represents each club's membership as a function of x, the number of students in the ski club.

Club	Ski	Chess	Model UN	Math
# kids	x	$\frac{x}{2}$	$\frac{x}{6}$	$\frac{x}{10}$

Since every club has a whole number of kids in it, the number of kids in the ski club, or x, must be evenly divisible by $\text{lcm}(2, 6, 10) = 30$. Additionally, there are fewer than 160 kids in the ski club. This table lists the possible roster sizes for the clubs as a result.

Ski	Chess	Model UN	Math
150	75	25	15
120	60	20	12
90	45	15	90
60	30	10	6
30	15	5	3

Given that everyone has a partner to play against at chess club practice, the number of students in that club must be even. Possible roster sizes are listed below.

Ski	Chess	Model UN	Math
120	60	20	12
60	30	10	6

The largest possible difference between the number of students in Model UN and the math team is $\boxed{8}$.

Round 2 - Algebra I

1. Solve for t : $\frac{1}{5}(2t+3)+\frac{3}{5}(8-3t)=-3$

Solution:

$$\begin{aligned}\frac{1}{5}(2t+3)+\frac{3}{5}(8-3t)&=-3\\ \frac{2}{5}t+\frac{3}{5}+\frac{24}{5}-\frac{9}{5}t&=-3\\ -\frac{7}{5}t&=-\frac{15}{5}-\frac{27}{5}\\ -7t&=-42\\ t&=\boxed{6}\end{aligned}$$

2. A zebra leaves a watering hole running at 60 km/h. After the zebra has been running for 90 seconds, a wild dog leaves the watering hole and chases after the zebra at 70 km/h. If both animals maintain their speed, how many minutes does it take the dog to catch up with the zebra after the dog started running?

Solution: The zebra, running at 60 km/h or 1 km/ min, will cover 1.5 kilometers in 90 seconds or 1.5 minutes or $\frac{1.5}{60}$ hours. Since the zebra and the wild dog will have traveled the same distance while running at different rates for a different amount of time, we construct a rate table to equate their distances.

Animal	Rate ($\frac{\text{km}}{\text{hr}}$)	Time (hr)	Distance (km)
Zebra	60	$t+\frac{1.5}{60}$	$60t+1.5$
Wild Dog	70	t	$70t$

Knowing the two distances are equal,

$$\begin{aligned}60t+1.5&=70t\\ 1.5&=10t\\ 9&=60t\\ \frac{9}{60}&=t\end{aligned}$$

With t representing $\frac{9}{60}$ of an hour, we determine that it takes $\frac{9}{60}\text{hr}\times\frac{60\text{ min}}{1\text{ hr}}=\boxed{9\text{ minutes}}$ for the wild dog to catch the zebra.

Alternative Method: Since the wild dog is running at 70 km/h, the wild dog will close the 1.5 km gap at a rate of $70-60=10\text{km/h}=\frac{1}{6}\text{km/ min}$. At this rate, the $\frac{9}{6}$ km gap will be closed in $\frac{9}{6}\div\frac{1}{6}=\boxed{9\text{ minutes}}$.

3. The sum of 22^2 and 19^2 equals the sum of another pair of two-digit numbers squared. Find the two numbers.

Solution: This problem states that there are two two-digit numbers, a and b, for which $22^2 + 19^2 = a^2 + b^2$. Intuitively, to change the numbers but keep the sum the same, either a and b are both between 22 and 19, or one of them is larger than 22 while the other is between 18 and 10 inclusive. Computing $22^2 = 484$ and $19^2 = 381$, their sum is 845. Letting $a > b$, check increasing squares of a for 23, 24, 25... and subtracting from 845.

a	a^2	$845 - a^2$	Square?
23	529	326	no
24	576	269	no
25	625	220	no
26	676	169	yes! - 13^2

The two numbers are $\boxed{\text{13 and 26}}$. You could also try subtracting increasing squares from 10 upward, but it would leave you with larger possible squares which are not always recognized.

Alternative Method: Combining the squares, factoring, distributing, and combining:

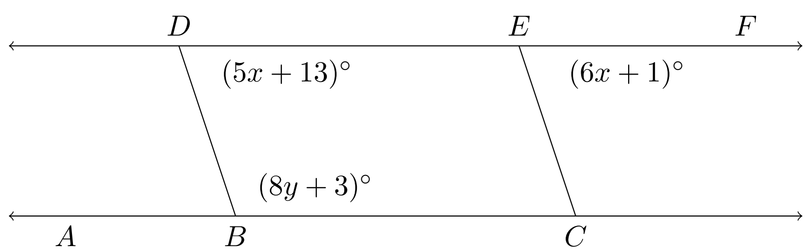
$$22^2 + 19^2 = 845 = 5 \cdot 169 = 5 \cdot 13^2 = 4 \cdot 13^2 + 13^2 = 2^2 \cdot 13^2 + 13^2 = 26^2 + 13^2.$$

Round 3 - Parallel Lines and Polygons

1. In the diagram below, $\overline{BD} \parallel \overline{CE}$. Find the values of x and y such that $\overline{BC} \parallel \overline{DE}$.

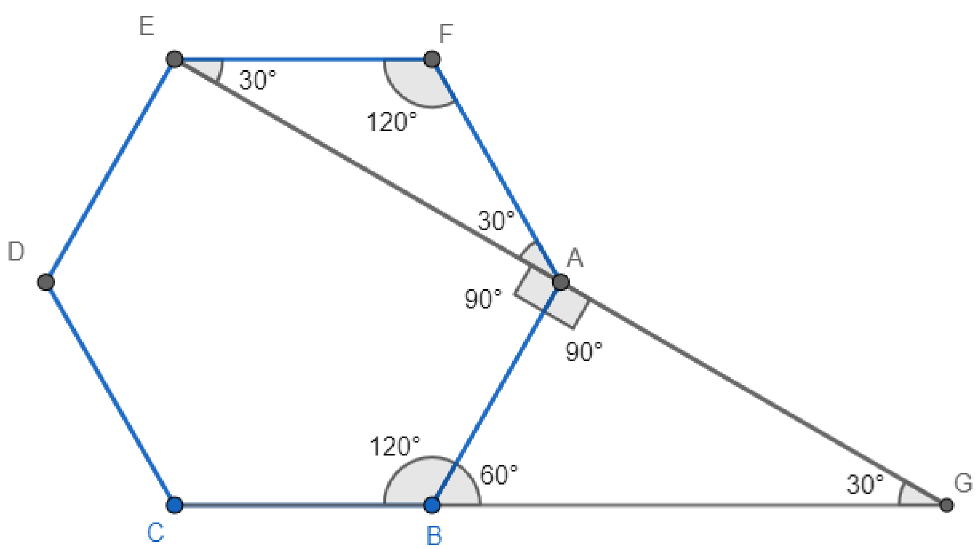
Solution: Since $\overline{BD} \parallel \overline{CE}$, we know that $\angle BDE \cong \angle CEF$ since they are corresponding and that their measures must be equal, so $5x + 13 = 6x + 1 \Rightarrow x = \boxed{12}$. To cause $\overline{BC} \parallel \overline{DE}$, $\angle BDE$ and $\angle DBC$ must be supplementary, so $5x + 13 + 8y + 3 = 180$. Letting $x = 12$ in this equation,

$$\begin{aligned} 60 + 13 + 8y + 3 &= 180 \\ 8y &= 104 \\ y &= \boxed{13} \end{aligned}$$



2. If ABCDEF is a regular hexagon, what acute angle is formed by the intersection of $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$? Express your answer in degrees.

Solution: Let $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$ intersect at point G. Each of the six interior angles of a hexagon is $\frac{(6-2)\cdot 180^\circ}{6} = 120^\circ$. This means that $\measuredangle EFA = \measuredangle FAB = \measuredangle ABC = 120^\circ$. Since ΔFEA is isosceles, then the base angles are equal so $\measuredangle FEA = \measuredangle FAE = 30^\circ$ each, leaving $\measuredangle EAB = 90^\circ = \measuredangle BAG$. Since exterior angles of a hexagon $\frac{360^\circ}{6} = 60^\circ$ each, $\measuredangle ABG = 60^\circ$, leaving $\measuredangle G = 180^\circ - 90^\circ - 60^\circ = \boxed{30^\circ}$.



3. Given:

   $\overline{BE} \parallel \overline{CD}$

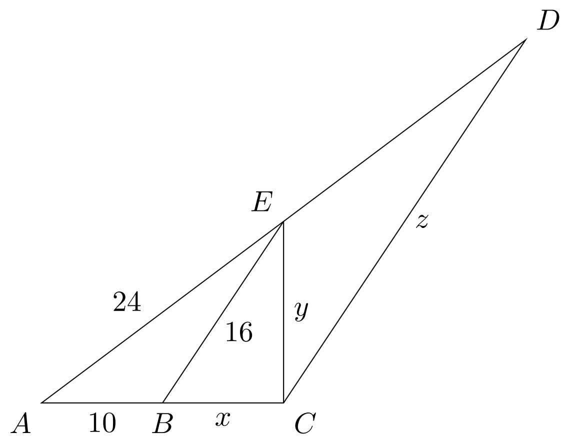
   $\overline{AC} \perp \overline{EC}$

   $AB = 10$

   $AE = 24$

   $BE = 16$

   Find CD.



Solution: After labeling the diagram, we find ourselves with two similar triangles of ΔABE and ΔACD. Since the triangles are similar, the following proportion must be true:

$$\frac{10}{16} = \frac{10 + BC}{CD}.$$

To solve for CD, we only need to identify BC. We recognize two right triangles (ΔACE and ΔBCE). Relating the lengths of their sides with the Pythagorean Theorem and knowing x and y are consistent in both equations, to find x and y we can solve the following system:

$$\begin{cases} (x+10)^2 + y^2 &= 24^2 \\ x^2 + y^2 &= 16^2 \end{cases}$$

Solving for x (or BC) via elimination:

$$\begin{aligned} (x + 10^2) - x^2 &= 24^2 - 16^2 \\ x^2 + 20x + 100 - x^2 &= 576 - 256 \\ 20x &= 220 \\ x &= 11 \end{aligned}$$

Substituting into our original proportion and solving for CD:

$$\begin{aligned} \frac{10}{16} &= \frac{10 + 11}{CD} \\ 10 \cdot CD &= 336 \\ CD &= \frac{336}{10} = \boxed{\frac{168}{5} = 33\frac{3}{5} = 33.6} \end{aligned}$$

Round 4 - Sequences and Series

1. Find the first term of the arithmetic sequence if the 5th term is 24 and the 9th term is 40.

Solution: The nth term, a_n, of an arithmetic sequence with first term a_1 and common difference d is defined by $a_n = a_1 + d(n-1)$. Therefore

$$24 = a_1 + 4d \qquad\qquad 40 = a_1 + 8d$$

Solving for a_1 in each equation we use the transitive property to find that

$$\begin{aligned} 24 - 4d &= 40 - 8d \\ 4d &= 16 \\ d &= 4 \end{aligned}$$

Plugging d back into either equation gives us $a_1 = \boxed{8}$.

Shortcut: Find d knowing that from the 5th term to the 9th term requires adding four differences to the 5th term. Solving $24 + 4d = 40$ gives $d = 4$ quite quickly.

2. Evaluate.

$$\sum_{k=1}^{11} 2^{k-3}$$

Solution: This is a finite geometric series with $a_1 = 256$, $r = \frac{1}{2}$, and $n = 11$.

$$S_{11} = \frac{a_1(a - r^{11})}{1 - r} = \frac{256(1 - \left(\frac{1}{2}\right)^{11})}{(1 - \frac{1}{2})} = \frac{256\left(1 - \frac{1}{2048}\right)}{\frac{1}{2}} = 2^8 \cdot \frac{2047}{2^{11}} \cdot 2 = \frac{2047}{4}$$

Alternative Method: We can also view this with $a_1 = \frac{1}{4}$ and $r = 2$. The sum of this series is

$$S_{11} = \frac{a_1(a - r^{11})}{1 - r} = \frac{\frac{1}{4}(1 - 2^{11})}{(1 - 2)} = \frac{\frac{1}{4}(-2047)}{-1} = \frac{2047}{4}$$

Alternative Method: Consider that

$$\sum_{k=1}^{11} 2^{k-3} = 2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} + 2^{-2}$$

Recall that

$$(x^n + x^{n-1} + \cdots x^2 + x^1 + 1) = \frac{x^{n+1} - 1}{x - 1}.$$

This means that

$$2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} + 2^{-2} = \frac{2^9 - 1}{2 - 1} + \frac{1}{2} + \frac{1}{4} = 511.75$$

The sum is $\boxed{\frac{2047}{4} = 511\frac{3}{4} = 511.75}$.

3. Solve for a.

$$\sum_{x=a}^{40}(2x-5) = 1428$$

Solution: This is a finite arithmetic series with $a_1 = 75$, $a_n = 2a - 5$, and $n = (40 - a) + 1 = 41 - a$.

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{(41-a)(75+(2a-5))}{2} = \frac{(41-a)(70+2a)}{2} = (41-a)(35+a) = 1428$$

Solving for a:

$$\begin{aligned}(41-a)(35+a) &= 1428\\ 1435 + 6a - a^2 &= 1428\\ 0 &= a^2 - 6a - 7\\ 0 &= (a-7)(a+1)\\ a = 7 &\text{ or } a = -1\end{aligned}$$

The two values of a that cause the equation to be true are $\boxed{7 \text{ or } -1}$. The What Goes Up Might Come Down Principle of Finite Arithmetic Series With $a_1 > 0$ and $d < 0$ gives us this interesting result.

Round 5 - Matrices and Systems of Equations

1. Find $A(B+C)$ if $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3 + 3 \cdot -1 & 2 \cdot -5 + 3 \cdot 5 \\ 1 \cdot 3 + 2 \cdot -1 & 1 \cdot -5 + 2 \cdot 5 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 3 & 5 \\ 1 & 5 \end{bmatrix}}$$

2. The expression below is equivalent to the identity matrix. Find k: $\begin{bmatrix} 2 & 1 & k \\ 3 & k & 4 \end{bmatrix} \begin{bmatrix} k & 4 \\ -2 & -1 \\ -3 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -11 \\ 13 & 4 \end{bmatrix}$

Solution: First, equate the expression with the identity matrix of the appropriate size. In this case, it will be a 2×2 matrix since the product of the first two matrices will be added to the third, so the product must also be a 2×2 matrix. Second, combine the three matrices. Finally, solve for k.

$$\begin{bmatrix} 2 & 1 & k \\ 3 & k & 4 \end{bmatrix} \begin{bmatrix} k & 4 \\ -2 & -1 \\ -3 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -11 \\ 13 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2k - 2 - 3k & 8 - 1 - 4k \\ 3k - 2k - 12 & 12 - k - 16 \end{bmatrix} + \begin{bmatrix} 2 & -11 \\ 13 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -k & -4 - 4k \\ k + 1 & -k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-k = 1 \qquad -4 - 4k = 0 \qquad k + 1 = 0 \qquad -k = 1$$
$$k = -1 \qquad k = -1 \qquad k = -1 \qquad k = -1$$

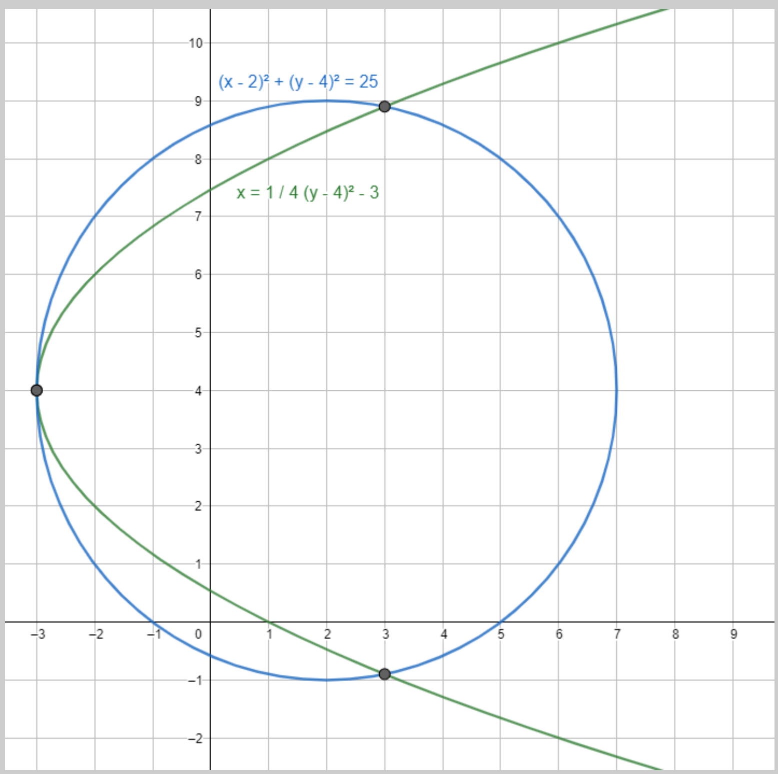
Solving any particular equation will give $\boxed{k = -1}$.

3. Circle C is centered at $O(2, 4)$. Parabola P has a vertex of $(-3, 4)$ and an x-intercept of 1. Find all three points of intersection of circle C and parabola P.

Solution: Given three points of intersection for a parabola and a circle, the vertex of the parabola must be internally tangent to the circle. Given this, we know that if the circle has center $(2, 4)$ and the parabola has vertex $(-3, 4)$, the circle must have a radius of 5 and an equation of $(x-2)^2 + (y-4)^2 = 25$. Additionally, for the parabola to be tangent to the circle, it must open right and be of the form $x = a(y-4)^2 - 3$, with a being the leading coefficient. Since we are given the sole x-intercept as $(1, 0)$ of the parabola, we can solve for a.

$$\begin{aligned} x &= a(y-4)^2 - 3 \\ 1 &= a(0-4)^2 - 3 \\ 4 &= 16a \\ \frac{1}{4} &= a \end{aligned}$$

The situation is graphed below.



To find the points of intersection, we must solve the following system:

$$\begin{cases} (x-2)^2 + (y-4)^2 &= 25 \\ x &= \frac{1}{4}(y-4)^2 - 3 \end{cases}$$

$$\begin{cases} (x-2)^2 + (y-4)^2 &= 25 \\ 4(x+3) &= (y-4)^2 \end{cases}$$

Solving through substitution:

$$\begin{aligned}(x-2)^2+4(x+3)&=25\\x^2-4x+4+4x+12&=25\\x^2&=9\\x&=\pm 3\end{aligned}$$

We already know one point of intersection where the parabola is internally tangent to the circle ($(-3,4)$). The other two points of intersection occur when $x=3$:

$$\begin{aligned}3&=\frac{1}{4}(y-4)^2-3\\24&=(y-4)^2\\\sqrt{24}&=|y-4|\\\pm 2\sqrt{6}&=y-4\\4\pm 2\sqrt{6}&=y\end{aligned}$$

The three points of intersection are $\boxed{(-3,4),\left(3,4+2\sqrt{6}\right)\text{ and }\left(3,4-2\sqrt{6}\right)}$

Team Round

1. Write the following as the ratio of two relatively prime integers: 0.4131313...

Solution: Let $N = 0.4131313...$

$$\begin{aligned} 100N &= 41.3131313\ldots \\ N &= 0.4131313\ldots \\ 100N - N = 99N &= 40.9 \\ N &= \frac{40.9}{99} = \boxed{\frac{409}{990}} \end{aligned}$$

Since $990 = 2 \cdot 3^2 \cdot 5 \cdot 11$ and 409 isn't divisible by 2, 3, 5, or 11, 409 and 990 are relatively prime.

2. Evaluate $3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$.

Solution:

$$= 3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{\frac{9}{2}}}} = 3 + \cfrac{1}{4 + \cfrac{1}{1 + \frac{2}{9}}} = 3 + \cfrac{1}{4 + \cfrac{1}{\frac{11}{9}}} = 3 + \cfrac{1}{4 + \frac{9}{11}} = 3 + \cfrac{1}{\frac{53}{11}} = \boxed{3\frac{11}{53} = \frac{170}{53}}$$

3. John, Paul, George, Ringo, and Stuart arrive in Hamburg, Germany which has only six hotels. If each person picks a random hotel to stay at, what is the probability that no two people from this group will go to the same hotel?

Solution: Consider the five men standing in a line: John, Paul, George, Ringo, and Stuart. John, the first in line, picks a hotel at random, but is guaranteed to pick a hotel that no one else in the group is at. The second man, Paul, picks a hotel at random, too, but there is a $\frac{1}{6}$ chance that he will pick the same hotel that John has picked, meaning there is a $\frac{5}{6}$ chance that he will pick a hotel that is different. Similarly, George's, Ringo's, and Stuart's chances of picking a hotel that none of the others in the group have picked before them are $\frac{4}{6}$, $\frac{3}{6}$, and $\frac{2}{6}$, respectively. These are all independent events and can be multiplied together to achieve the probability that no two men in the group will pick the same hotel.

$$1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} = \frac{2^3 \cdot 3 \cdot 5}{2^4 \cdot 3^4} = \frac{5}{2 \cdot 3^3} = \boxed{\frac{5}{54}}$$

4. A crew can row a 2 kilometer course downriver (with the current) in just 6 minutes, but it takes the crew 15 minutes to row the same distance upriver (against the current). Find the rate of the crew in still water and the rate of the current, express those rates in kilometers per hour.

Solution: Going downriver, the crew is moving

$$\frac{2\text{ km}}{6\text{ min}} \cdot \frac{10}{10} = \frac{20\text{ km}}{60\text{ min}} = 20\frac{\text{km}}{\text{hr}}.$$

Going upriver, the crew is moving

$$\frac{2\text{ km}}{15\text{ min}} \cdot \frac{4}{4} = \frac{8\text{ km}}{60\text{ min}} = 8\frac{\text{km}}{\text{hr}}.$$

If R is the speed of the rowers in still water and C is the speed of the current, we need to solve the system

$$\begin{cases} R + C &= 20 \\ R - C &= 8 \end{cases}$$

Using elimination, $2R = 28 \Rightarrow R = 14$. From this, we know $C = 6$. This means the crew can row $\boxed{\text{14 km / hr in still water}}$ and the rate of the current is $\boxed{\text{6 km / hr}}$.

5. Given parallelogram ABCD, $\measuredangle E = 62°$, $\measuredangle BFE = 32°$, and $\measuredangle C = 68°$, find $\measuredangle ADE$.

Solution:

- First, we know that $ABCD$ is a parallelogram, so opposite sides are parallel to one another, opposite angles are congruent, and adjacent angles are supplementary.
- Draw segment $\overline{GE}$ so it is parallel to $\overline{AB}$ and $\overline{DC}$. From this, we know $\measuredangle BFE = \measuredangle FEG$ are alternate interior, and since $\overline{BF} \parallel \overline{GE}$ then their measures are equal.
- We also know that $\measuredangle FEG + \measuredangle GED = 62°$, so $\measuredangle GED = 30°$. We also know $\measuredangle GED = \measuredangle EDC$ are alternate interior, and since $\overline{GE} \parallel \overline{DC}$ then their measures are equal.

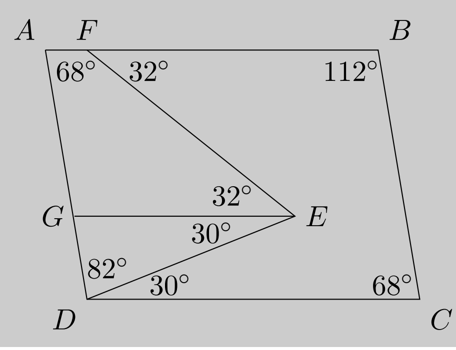
Finally, since $ABCD$ is parallelogram, we know:

$$\measuredangle ADC + \measuredangle DCB = 180°$$

$$\measuredangle ADE + \measuredangle EDC + \measuredangle DCB = 180°$$

$$\measuredangle ADE + 30° + 68° = 180°$$

$$\boxed{\measuredangle ADE = 82°}$$



6. Evaluate: $\displaystyle\sum_{k=1}^{18}\left(\frac{1}{k}-\frac{1}{k+2}\right)$.

Solution:

$$\begin{aligned}\sum_{k=1}^{18}\left(\frac{1}{k}-\frac{1}{k+2}\right)=\sum_{k=1}^{18}\frac{1}{k}-\sum_{k=1}^{18}\frac{1}{k+2}&=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}\right)\\&\quad-\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}\right)\\&=\frac{1}{1}+\frac{1}{2}-\frac{1}{19}-\frac{1}{20}\\&=\frac{380}{380}+\frac{190}{380}-\frac{20}{380}-\frac{19}{380}\\&=\boxed{\frac{531}{380}=1\frac{151}{380}}\end{aligned}$$

Since $380=2^2\cdot 5\cdot 19$ and $531=9\cdot 59=3^2\cdot 59$, then 531 and 380 are relatively prime.

7. Solve for x.

$$\begin{vmatrix}x+4 & 1 & 1\\ 2 & 2 & 2\\ 3 & x-2 & 3\end{vmatrix}=0$$

Solution: Since $\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}=aei+bfg+cdh-gec-hfa-idb$,

$$\begin{vmatrix}x+4 & 1 & 1\\ 2 & 2 & 2\\ 3 & x-2 & 3\end{vmatrix}=6(x+4)+6+2(x-2)-6-2(x-2)(x+4)-6=0$$

$$6x+24+2x-4-2x^2-4x+16-6=0$$
$$-2x^2+4x+30=0$$
$$x^2-2x-15=0$$
$$(x-5)(x+3)=0$$
$$\boxed{x=\{-3,5\}}$$

Alternative Method: Using matrix column operations, subtract the third column from the first and the third column from the second to obtain

$$\begin{vmatrix}x+3 & 0 & 1\\ 0 & 0 & 2\\ 0 & x-5 & 3\end{vmatrix}=0+0+0-0-2(x+3)(x-5)-0=0 \quad\Rightarrow\quad -2(x+3)(x-5)=0 \quad\Rightarrow\quad x=\{-3,5\}$$

8. In the expansion of $(x^2-2)^{15}$, determine the sum of the coefficients.

Solution: Let's begin the brute force trek!

$$\begin{aligned}(x^2-2)^{15} = &\binom{15}{0}(x^2)^{15}(-2)^0 + \binom{15}{1}(x^2)^{14}(-2)^1 + \binom{15}{2}(x^2)^{13}(-2)^2 + \binom{15}{3}(x^2)^{12}(-2)^3 \\ &+ \binom{15}{4}(x^2)^{11}(-2)^4 + \binom{5}{6}(x^2)^{10}(-2)^5 + \binom{6}{7}(x^2)^{9}(-2)^6 + \binom{7}{8}(x^2)^{8}(-2)^7 \\ &+ \binom{15}{8}(x^2)^{7}(-2)^8 + \binom{15}{9}(x^2)^{6}(-2)^9 + \binom{15}{10}(x^2)^{5}(-2)^{10} + \binom{15}{11}(x^2)^{4}(-2)^{11} \\ &+ \binom{15}{12}(x^2)^{3}(-2)^{12} + \binom{15}{13}(x^2)^{2}(-2)^{13} + \binom{15}{14}(x^2)^{1}(-2)^{14} + \binom{15}{15}(x^2)^{0}(-2)^{15}\end{aligned}$$

This does not look fun. Let's change strategies and generalize. We know that

$$(x^2-2)^{15} = a_1x^{30} + a_2x^{28} + a_3x^{26} + \cdots + a_{14}x^4 + a_{15}x^2 + a_{16}$$

and we want to add up all the values of a_n Let $x=1$. We find

$$(1-2)^{15} = a_1 + a_2 + a_3 + \cdots + a_{14} + a_{15} + a_{16} = \boxed{-1}$$

9. Determine the value of $384,958^2 - 384,957^2$.

Solution: For two consecutive integers, n and $n+1$, we find the difference of their squares to be

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$$

In this case, $n = 384,957$ and $2 \cdot (384,957) + 1 = \boxed{769,915}$

Alternative Method: Look for patterns in differences of consecutive squares.

$$2^2 - 1^2 = 3 = 2\cdot 2 - 1 = 2 \cdot 1 + 1$$

$$3^2 - 2^2 = 5 = 2\cdot 3 - 1 = 2 \cdot 2 + 1$$

$$4^2 - 3^2 = 7 = 2\cdot 4 - 1 = 2 \cdot 3 + 1$$

$$5^2 - 4^2 = 9 = 2\cdot 5 - 1 = 2 \cdot 4 + 1$$

Double the larger number and subtract one, or double the smaller number and add one.