Worcester County Mathematics League

Varsity Meet 4 - February 28, 2024

Round 1 - Elementary Number Theory

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Write the decimal number 11 in base 4.

2. Find the least positive integer n such that $882 \cdot n$ is a perfect cube.

3. Find the largest integer that divides $x^2\left(x^2-1\right)\left(x^2-4\right)$ for any positive integer x. Note that *every* integer divides zero.

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. ________________

(3 pts) 3. ________________

Oxford, Worc. Acad., Anonymous

Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the number halfway between $\frac{1}{2}$ and $\frac{3}{5}$.

2. Adrian's Pet Shop sells guinea pigs, white mice, and white rats. A rat costs twice as much as a mouse, and a guinea pig costs twice as much as a rat. Mr. Balboa buys 3 mice, 5 rats and 7 guinea pigs. Mr. Creed buys 5 mice, 7 rats and 3 guinea pigs. If Mr. Creed spends $65 less than Mr. Balboa, how much does a mouse cost?

3. Let m and n be integers such that $m > 0$ and

$$2m^2 - 2mn + n^2 = 289.$$

Let $s = m + n$ for a particular such ordered pair (m, n). Find the sum of all possible values of s.

ANSWERS

(1 pt) 1. ____________

(2 pts) 2. $ ____________

(3 pts) 3. ____________

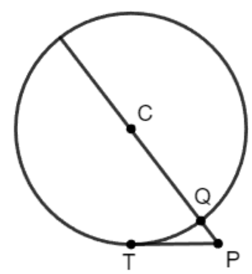
Leicester, Hudson, Algonquin/QSC

Worcester County Mathematics League
Varsity Meet 4 - February 28, 2024
Round 3 - Geometry

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Given a circle with radius 15cm, find the area of a sector with a central angle of 120°, in cm^2.

2. Given that $\overline{PT}$ is tangent to $\odot C$ (circle C) at T as shown at right, where $\overline{PC}$ intersects $\odot C$ at Q , $QP = 1$, and $TP = 5$, find the area of $\odot C$ expressed as a multiple of π.



3. Given five line segments of lengths $1, 2, \frac{5}{2}, 3$ and 4, find the sum of the perimeters of all scalene triangles that can be formed with three of these line segments.

ANSWERS

(1 pt) 1. ____________

(2 pts) 2. ____________

(3 pts) 3. ____________

Grafton, Worc. Acad., Monty Tech

Worcester County Mathematics League
Varsity Meet 4 - February 28, 2024
Round 4 - Logs, Exponents, and Radicals

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Evaluate the following expression:

$$\left(\left(64^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{2}$$

2. The expression below can be expressed in simplest radical form as $m\sqrt{n}$. Find the ordered pair of integers (m, n).

$$\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}$$

3. Find all ordered pairs (x, y) that satisfy the following system of equations:

$$\log_3\left(3^{x-4}3^{4y}\right) = (\sqrt{3})^{\log_3 16xy}$$
$$\sqrt{x}+2\sqrt{y}=8$$

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. $(m, n) =$ (______________)

(3 pts) 3. $(x, y) \in$ { ____________________________ }

Tahanto/QSC, Clinton, Worc. Acad./QSC

Worcester County Mathematics League
Varsity Meet 4 - February 28, 2024
Round 5 - Trigonometry

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the value of:
$$\cos^2(15°) + \cos^2(25°) + \cos^2(65°) + \cos^2(75°)$$

2. Simplify the following expression:
$$\frac{\sin^3 x + \cos^3 x}{1 - \sin x \cos x}$$

3. Find all values of θ, where $0 \leq \theta < 360°$, that solve the following equation shown below. Express your answer in °.
$$\cos\theta = \cos(105°) + \cos(225°)$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $\theta \in \{$ ____________________ $°\}$

Algonquin, Doherty, Westborough

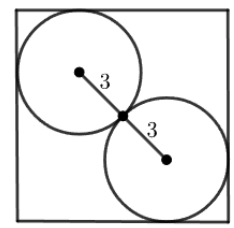
Worcester County Mathematics League
Varsity Meet 4 - February 28, 2024
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. How many three digit positive integers s have the property that $s^2 - 1$ is a multiple of 3?

2. Given: $\dfrac{a-b}{c-b} = -4$; find $\dfrac{a-c}{b-c}$.

3. Two circles, externally tangent and each with radius 3, are inscribed in a square as shown at right. What is the perimeter of the square?



4. Solve the following equation for Real numbers x. Give your answer in the form $\log_b a$ where a and b are integers and b is as small as possible.

$$3^x - 4 - 5 \cdot 3^{-x} = 0$$

5. The period of a periodic function $f(x)$ is the smallest value T such that $f(x) = f(x - T)$ for all values of x. Find the period of $g(x) = \frac{3}{4}\cos\frac{2}{3}x$, where x is measured in radians. Express your answer in radians as a multiple of π.

6. Lilly divides her favorite number by a smaller integer d resulting in a remainder of 24. She then divides twice her number by d resulting in a remainder of 11. Find d.

7. Simplify the following expression:

$$(x+y)^3 - \left(x^3 + y^3\right)$$

8. Given positive integers a and b, where $a^2 + b^2 = 23ab$, find the value of the following expression expressed as a single natural logarithm (ln):

$$\ln(a+b) - \frac{\ln a + \ln b}{2}$$

9. Given functions $f(x) = \sin 3x$ and $g(x) = 2\sin x$, find all values of x (expressed in radians) where $0 \le x < 2\pi$, such that $f(x) = g(x)$.

Hudson, Worc. Acad., Wachusett, Leicester, Shrewsbury, Sutton, Southbridge, Anon., Quaboag

Worcester County Mathematics League
Varsity Meet 4 - February 28, 2024
Team Round Answer Sheet

ANSWERS

1. ______________________

2. ______________________

3. ______________________

4. $x =$ ______________________

5. $T =$ ______________________

6. ______________________

7. ______________________

8. ______________________

9. $x \in \{$ ____ $\}$

Worcester County Mathematics League
Varsity Meet 4 - February 28, 2024
Answer Key

Round 1 - Elementary Number Theory

1. 23 or 23_4
2. 84
3. 360

Round 2 - Algebra I

1. $\frac{11}{20}$ or 0.55 or .55
2. 6.50 or \$6.50 but not $\frac{13}{2}$ nor $6\frac{1}{2}$
3. 126

Round 3 - Geometry

1. 75π
2. 144π
3. $\frac{93}{2}$ or $46\frac{1}{2}$ or 46.5

.

Round 4 - Logs, Exponents, and Radicals

1. 4
2. $(2, 2)$
3. $\left(9, \frac{25}{4}\right), \left(25, \frac{9}{4}\right)$ need both ordered pairs listed in either order (but each ordered pair exactly ordered). Also acceptable: $\left(9, 6\frac{1}{4}\right), \left(25, 2\frac{1}{4}\right)$ or $(9, 6.25), (25, 2.25)$

Round 5 - Trigonometry

1. 2
2. $\sin x + \cos x$ (either order)
3. $165°, 195°$ (need both, either order)

Team Round

1. 600
2. 5
3. $24 + 12\sqrt{2}$ or $12\sqrt{2} + 24$
4. $\log_3 5$
5. 3π
6. 37
7. $3x^2y + 3xy^2$ or $3xy(x + y)$ (Order doesn't matter)
8. $\ln 5$
9. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ (Need all six, any order)

Round 1 - Elementary Number Theory

1. Write the decimal number 11 in base 4.

Solution: Note that a base four number with three digits is converted to a base 10 number by summing powers of 4:

$$\begin{aligned}(abc)_4 &= a \cdot 4^2 + b \cdot 4^1 + c \cdot 4^0 \\ &= 16a + 4b + c\end{aligned}$$

Instead of the "hundreds place", the "tens place" and the "ones place" in a decimal representation, the base 4 number has a "sixteens place", a "fours place" and a "ones place". Since $11 < 16$, then $a = 0$; there is no number in the sixteens place nor in any higher place, and 11 is therefore a two digit number in base 4. Then divide 11 by 4, resulting in 2 with a remainder of 3. Since $11 = 2 \cdot 4 + 3$, the base four representation is $\boxed{23_4}$.

Alternative Solution: The most direct method to convert a decimal number (in base 10) to its representation in another base, say b, is to divide the number by b and record the remainder, repeating the operation at each step on the quotient from the previous step until the dividend is zero. The base b representation is found as the remainders in reverse order. Thus, $11 \div 4 = 2\,\text{R}\,3$ (2 with remainder 3), and $2 \div 4 = 0\,\text{R}\,2$. The remainders in order are $3, 2$, and the remainders in reverse order are $2, 3$. The base 4 representation is therefore 23_4.

2. Find the least positive integer n such that $882 \cdot n$ is a perfect cube.

Solution: First find the prime factorization: $882 = 2 \cdot 441 = 2 \cdot 9 \cdot 49 = 2 \cdot 3^2 \cdot 7^2$. Note that the prime factorization of a perfect cube will have all its primes taken to a power of 3 or a multiple of 3. That is because the cube root of a product equals the product of the individual cube roots. In this case, $\sqrt[3]{2^3 \cdot 3^3 \cdot 7^3} = \sqrt[3]{2^3} \cdot \sqrt[3]{3^3} \cdot \sqrt[3]{7^3} = 2 \cdot 3 \cdot 7$. Therefore if $882 \cdot n = 2^3 \cdot 3^3 \cdot 7^3$, it will be a perfect cube, and:

$$\begin{aligned}n &= \frac{2^3 \cdot 3^3 \cdot 7^3}{882} \\ &= \frac{2^3 \cdot 3^3 \cdot 7^3}{2^1 \cdot 3^2 \cdot 7^2} \\ &= 2^{3-1} \cdot 3^{3-2} \cdot 7^{3-2} \\ &= 2^2 \cdot 3 \cdot 7 \\ &= 4 \cdot 21 = \boxed{84}.\end{aligned}$$

3. Find the largest integer that divides $x^2(x^2-1)(x^2-4)$ for any positive integer x. Note that *every* integer divides zero.

Solution: First, name the given polynomial $p(x)$ and then factor it:

$$\begin{aligned} p(x) &= x^2\left(x^2-1\right)\left(x^2-4\right) \\ &= x^2(x-1)(x+1)(x-2)(x+2) \\ &= (x-2)(x-1)x^2(x+1)(x+2) \end{aligned}$$

Note that, for any given x, $p(x)$ will have five consecutive integers that are factors. The first case of interest is $x=3$, because $p(1)=p(2)=0$, and every integer divides zero. Then the prime factorization of $p(3)$ is:

$$p(3) = 1\cdot 2\cdot 3^2\cdot 4\cdot 5 = 2\cdot 3^2\cdot 4\cdot 5 = 360$$

Therefore 360 is the largest integer that divides $p(3)$, so the largest integer that divides $p(x)$ for all $x>0$ is no larger than 360.

Note that 5 will divide $p(x)$ for any value of x because every fifth integer is divisible by 5.

Next, note that for any x, 3 will divide either $(x-2)$, $(x-1)$, or x. In the first case, 3 also divides $(x+1)$, in the second case 3 also divides $(x+2)$, and in the third case 3^2 divides x^2. In all three cases, 3^2 will divide $p(x)$.

Finally, note that when x is odd, both $(x-1)$ and $(x+1)$ will be even, and one of these two numbers will be divisible by 4, since every other even number is divisible by 4. Thus, $2\cdot 4$ must divide $p(x)$ for all x, or in other words $2\cdot 4$ is a factor of $p(x)$. If x is even, $p(x)$ will have more than three factors of 2, as $(x-2), x$ and $(x+2)$ will each be even.

Combining these facts, it is clear that $2\cdot 3^2\cdot 4\cdot 5$ must divide $p(x)$ for all $x>0$, and the answer is $\boxed{360}$.

Round 2 - Algebra I

1. Find the number halfway between $\frac{1}{2}$ and $\frac{3}{5}$.

Solution: Note that the number halfway between two numbers is their average. That is, if $b > a$, then $a + \frac{b-a}{2} = \frac{a+b}{2} = b - \frac{b-a}{2}$. Therefore $x = \dfrac{\frac{1}{2}+\frac{3}{5}}{2} = \dfrac{\frac{5}{10}+\frac{6}{10}}{2} = \dfrac{\frac{11}{10}}{2} = \boxed{\dfrac{11}{20}}$.

2. Adrian's Pet Shop sells guinea pigs, white mice, and white rats. A rat costs twice as much as a mouse, and a guinea pig costs twice as much as a rat. Mr. Balboa buys 3 mice, 5 rats and 7 guinea pigs. Mr. Creed buys 5 mice, 7 rats and 3 guinea pigs. If Mr. Creed spends \$65 less than Mr. Balboa, how much does a mouse cost?

Solution: Let m be the cost of one mouse. Then a rat costs $2m$ and a guinea pig costs $2(2m) = 4m$. Let B be amount that Mr. Balboa spends and C be the amount that Mr. Creed spends. Then $B = 7(4m) + 5(2m) + 3m = 28m + 10m + 3m = 41m$. And $C = 3(4m) + 7(2m) + 5m = 12m + 14m + 5m = 31m$. Then $B - C = 41m - 31m = 10m = \$65$ and $m = \frac{65}{10} = \boxed{\$6.50}$.

3. Let m and n be integers such that $m > 0$ and

$$2m^2 - 2mn + n^2 = 289.$$

If $s = m + n$ for a particular such ordered pair (m, n), then find the sum of all possible values of s.

Solution: First, note that $289 = 17^2$. Next, apply the binomial square identity to show that the left hand side of the equation is the sum of two squares:

$$\begin{aligned} 2m^2 - 2mn + n^2 &= m^2 + m^2 - 2mn + n^2 \\ &= m^2 + (m-n)^2 = 289 = 17^2 \end{aligned}$$

Note that $17^2 + 0^2 = 17^2$ and the equation is satisfied if $(m-n)^2 = 0$ and $m^2 = 17^2$. One ordered pair is $(17, 17)$, but not $(-17, -17)$, $(0, -17)$ nor $(0, 17)$ because of the restriction that $m > 0$. Then $s = 17 + 17 = 34$ for this pair.

Note also that $8, 15, 17$ is a Pythagorean triple, where $8^2 + 15^2 = 17^2$, and that this triple is the only way to express 17^2 as the sum of two squares. The other possible values of m are therefore 8 and 15.

If $m = 8$, then $(m-n)^2 = (8-n)^2 = 15^2$, and $8 - n = \pm 15$. Solve for $n = 8 \mp 15$ and two more ordered pairs are $(8, 23)$ and $(8, -7)$. The values for s are $8 + 23 = 31$ and $8 + -7 = 1$.

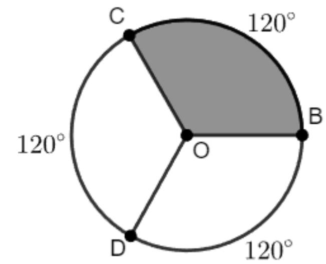
If $m = 15$, then $(m-n)^2 = (15-n)^2 = 8^2$, and $15 - n = \pm 8$. Solve for $n = 15 \mp 8$ and the last two ordered pairs are $(15, 7)$ and $(15, 23)$. The values for s are $15 + 7 = 22$ and $15 + 23 = 38$.

Finally, sum over all possible values of s: $34 + 31 + 1 + 22 + 38 = 34 + 32 + 60 = \boxed{126}$.

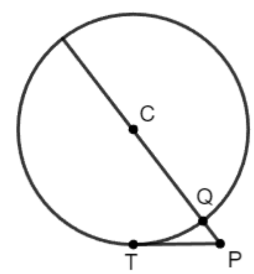
Round 3 - Geometry

1. Given a circle with radius 15cm, find the area, measured in cm^2, of a sector with a central angle of $120°$.

Solution: It may help to draw a circle such as $\odot O$ with shaded sector BOC measuring $120°$, as shown at right. Recall that if a circle is divided into arcs, the sum of the measures of the arcs is $360°$, and the same is true of their central angles. Note that $120 = 360 \div 3$, so that the area of sector BOC is one third of the area of $\odot O$. This fact is illustrated in the figure by drawing radius $\overline{OD}$ such that $m\angle COD = m\angle DOB = 120°$ and noting that sectors BOC, COD, and DOB are congruent, with equal measures.
The radius of $\odot O$ is $r = 15$cm, as given. Apply the formula for the area of the circle: $\pi r^2 = \pi 15^2 = 225\pi\ \text{cm}^2$. The area of the sector is therefore $225\pi \div 3 = \boxed{75\pi}\ \text{cm}^2$.



2. Given that $\overline{PT}$ is tangent to $\odot C$ (circle C) at T as shown at right, where $\overline{PC}$ intersects $\odot C$ at Q , $QP = 1$, and $TP = 5$, find the area of $\odot C$ expressed as a multiple of π.

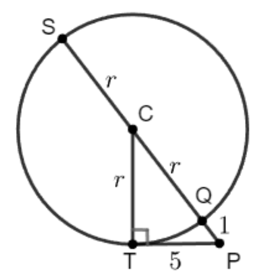


Solution: The area of a circle is equal to πr^2, where r is the radius of the circle. Thus, find r to find the area.
Note that the radius drawn to the point of tangency of a circle is perpendicular to the tangent. Draw radius $\overline{CT}$ and recognize the right triangle ΔCTP as shown at right. Let r be the radius of $\odot C$, so that $CT = CQ = r$, as labeled on the figure.
Now apply the Pythagorean Theorem to ΔCTP:

$$\begin{aligned} r^2 + 5^2 &= (r+1)^2 \\ r^2 + 25 &= r^2 + 2r + 1 \\ 25 - 1 &= 2r \end{aligned}$$

and $2r = 24$, or $r = 12$. Therefore the area of the circle is $\pi\left(12^2\right) = \boxed{144\pi}$.

Alternative solution: Recall that for any secant and tangent drawn from a point outside a given circle, the product of the lengths of the secant and its secant segment is equal to the square of the tangent segment length. In this case, $SP \cdot QP = PT^2$, $(2r+1) \cdot 1 = 5^2$, and $r = 12$ as above.



3. Given five line segments of lengths $1, 2, \frac{5}{2}, 3$ and 4, find the sum of the perimeters of all scalene triangles that can be formed with three of these line segments.

Solution: A scalene triangle is a triangle with three distinct side lengths. Begin by counting the number of ways to choose three of the five lengths: $\binom{5}{3} = \frac{5!}{(5-3)!3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$.

Next, recall the Triangle Inequality, which states that the lengths of any two sides of a triangle must be strictly greater than the length of the third side. In practice, it is sufficient to compare the sum of the two smaller lengths with the largest length. Note that the triplets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2.5, 4\}$, and $\{1, 3, 4\}$ do *not* satisfy the triangle inequality. Then calculate the perimeters of the triangles formed by the other 6 combinations of the five lengths:

$$1 + 2 + 2.5 = 5.5$$
$$1 + 2.5 + 3 = 6.5$$
$$2 + 2.5 + 3 = 7.5$$
$$2 + 2.5 + 4 = 8.5$$
$$2 + 3 + 4 = 9$$
$$2.5 + 3 + 4 = 9.5$$

Finally, the sum of these perimeters is $5.5 + 6.5 + 7.5 + 8.5 + 9 + 9.5 = 12 + 16 + 18.5 = \boxed{46.5}$.

Round 4 - Logs, Exponents, and Radicals

1. Evaluate the following expression:

$$\left(\left(64^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^2$$

Solution: Simplify the given expression using the rule of exponents: $(a^b)^c = a^{b\cdot c}$

$$\left(\left(64^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^2 = \left(64^{\frac{1}{2}}\right)^{\frac{2}{3}} = 64^{\frac{1}{2}\cdot\frac{2}{3}} = 64^{\frac{1}{3}}$$

Now $64^{\frac{1}{3}}$ is the same as $\sqrt[3]{64} = \boxed{4}$.

Alternatively, note that $64^{\frac{1}{2}} = \sqrt{64} = 8$, that $\left(64^{\frac{1}{2}}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ and that $\left(\left(64^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^2 = 2^2 = 4$.

2. The expression below can be expressed in simplest radical form as $m\sqrt{n}$. Find the ordered pair of integers (m, n).

$$\sqrt{5+2\sqrt{6}} - \sqrt{5-2\sqrt{6}}$$

Solution: Observe that $\left(5+2\sqrt{6}\right)$ and $\left(5-2\sqrt{6}\right)$ form a conjugate pair, and that the product of these two expressions will simplify to an integer. Proceeding, let $x = \sqrt{5+2\sqrt{6}} - \sqrt{5-2\sqrt{6}}$ and calculate x^2:

$$\begin{aligned} x^2 &= \left(\sqrt{5+2\sqrt{6}} - \sqrt{5-2\sqrt{6}}\right)^2 \\ &= 5+2\sqrt{6} - 2\sqrt{5+2\sqrt{6}}\sqrt{5-2\sqrt{6}} + 5 - 2\sqrt{6} \\ &= 10 - 2\sqrt{\left(5+2\sqrt{6}\right)\left(5-2\sqrt{6}\right)} \end{aligned}$$

Now apply the difference of squares identity in reverse: $\sqrt{(5+2\sqrt{6})(5-2\sqrt{6})} = \sqrt{5^2-(2\sqrt{6})^2} = \sqrt{25-24} = \sqrt{1} = 1$.

So $x^2 = 10 - 2\sqrt{5+2\sqrt{6}}\sqrt{5-2\sqrt{6}} = 10 - 2 = 8$ and $x = 2\sqrt{2}$. Therefore $(m, n) = \boxed{(2, 2)}$.

3. Find all ordered pairs (x, y) that satisfy the following system of equations:

$$\log_3\left(3^{x-4}3^{4y}\right) = (\sqrt{3})^{\log_3 16xy}$$
$$\sqrt{x} + 2\sqrt{y} = 8$$

Solution: First simplify the top equation using the properties of logs and exponents, starting with the left hand side (LHS) of this equation:

$$\begin{aligned}\log_3\left(3^{x-4}3^{4y}\right) &= \log_3 3^{x-4+4y}\\ &= x-4+4y\end{aligned}$$

(because $b^x b^y = b^{(x+y)}$ and $\log_b b^x = x$) and continuing with the right hand side (RHS) of the equation:

$$\begin{aligned}\left(\sqrt{3}\right)^{\log_3 16xy} &= \sqrt{3^{\log_3 16xy}}\\ &= \sqrt{16xy} = 4\sqrt{xy}\end{aligned}$$

(because $(\sqrt{a})^x = \sqrt{a^x}$ and $b^{log_b x} = x$). Equate the simplified expressions for the LHS and RHS: $x - 4 + 4y = 4\sqrt{xy}$, or $x + 4y - 4\sqrt{x}\sqrt{y} = 4$. Now let $u = \sqrt{x}$ and $v = \sqrt{y}$, so that $x = u^2$ and $y = v^2$. Then the first equation is equivalent to $u^2 + 4v^2 - 4uv = (u-2v)^2 = 4$ and $u - 2v = \pm 2$, and the original system of equations is equivalent to the following system expressed in u and v:

$$u - 2v = \pm 2$$
$$u + 2v = 8$$

Add the two equations: $2u = 8 \pm 2$, or $u = 4 \pm 1$, so $u = 3$ or $u = 5$. Then $2v = 8 - u$, or $v = 4 - \frac{u}{2}$ and $(u, v) = (3, \frac{5}{2})$ or $(5, \frac{3}{2})$. Finally, square the values for u and v and $(x, y) = \boxed{\left(9, \frac{25}{4}\right) \text{ or } \left(25, \frac{9}{4}\right)}$.

Alternative solution: Start with the simplified expressions for the LHS and RHS of the first equation:

$$x + 4y - 4\sqrt{xy} = 4.$$

Then square the second equation:

$$x + 4\sqrt{xy} + 4y = 64.$$

Add these two equations and $2x + 8y = 68$, or $x + 4y = 34$. Subtract the first equation from the second and $8\sqrt{xy} = 60$, so $\sqrt{xy} = \frac{15}{2}$ and $xy = \frac{225}{4}$.

Substitute $y = \frac{1}{4}(34 - x)$ into $xy = \frac{225}{4}$ to yield $x \cdot \frac{1}{4}(34 - x) = \frac{225}{4}$, so $34x - x^2 = 225$, or $x^2 - 34x + 225 = 0$.

Apply the quadratic formula and $x = \frac{34 \pm \sqrt{34^2 - 4 \cdot 225}}{2} = \frac{34 \pm 2\sqrt{289 - 225}}{2} = 17 \pm \sqrt{64} = 17 \pm 8$. Thus $x = 9$ or $x = 25$, and since $y = \frac{1}{4}(34 - x)$, the solutions are $(9, \frac{25}{4})$ and $(25, \frac{9}{4})$.

Round 5 - Trigonometry

1. Find the value of:

$$\cos^2(15°) + \cos^2(25°) + \cos^2(65°) + \cos^2(75°)$$

Solution: Recall two trigonometric identities:

$$\cos\theta = \sin(90° - \theta) \quad \text{(for } \theta \text{ measured in degrees)}$$

and

$$\sin^2\theta + \cos^2\theta = 1.$$

The two identities can be explained using the triangle trigonometry relations $\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$ and $\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$, where a is the length of the leg opposite angle α, b is the length of the leg opposite the complementary angle β, and c is the length of the hypotenuse. For the first identity, the two acute angles of a right triangle are complementary and their measures add to 90°. The adjacent side for α is the same side as the opposite side for the complementary angle $\beta = 90° - \alpha$. The second identity results from the Pythagorean Theorem: divide $a^2 + b^2 = c^2$ by c^2.

Apply these two identities in sequence to evaluate the given expression:

$$\begin{aligned}\cos^2 15° + \cos^2 25° + \cos^2 65° + \cos^2 75° &= \cos^2 15° + \cos^2 25° + \sin^2 25° + \sin^2 15° \\ &= \sin^2 15° + \cos^2 15° + \sin^2 25° + \cos^2 25° \\ &= 1 + 1 = \boxed{2}\end{aligned}$$

2. Simplify the following expression:

$$\frac{\sin^3 x + \cos^3 x}{1 - \sin x \cos x}$$

Solution: Recall that $\sin^2 x + \cos^2 x = 1$, so it follows that $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$. Substitute these into the numerator of the original expression, expand, group, and factor:

$$\begin{aligned}\sin^3 x + \cos^3 x &= \sin x(1 - \cos^2 x) + \cos x(1 - \sin^2 x) \\ &= \sin(x) - \sin x \cos^2 x + \cos x - \sin^2 x \cos x \\ &= (\sin(x) + \cos x) - (\sin x \cos^2 x + \sin^2 x \cos x) \\ &= (\sin(x) + \cos x) - (\cos x + \sin x)(\sin x \cos x) \\ &= (\sin(x) + \cos x)(1 - \sin x \cos x)\end{aligned}$$

Thus, the desired quotient is simply:

$$\frac{(\sin x + \cos x)(1 - \sin x \cos x)}{1 - \sin x \cos x} = \boxed{\sin x + \cos x}$$

Alternative solution: Apply the sum of cubes factorization ($a^3 + b^3 = (a+b)(a^2 - ab + b^2)$) to the numerator:

$$\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x) = (\sin x + \cos x)(1 - \sin x \cos x)$$

and cancel the common term with the denominator as above.

3. Solve the following expression for θ, where $0 \le \theta < 360°$. Express your answer in °.

$$\cos\theta = \cos(105°) + \cos(225°)$$

Solution: Note that there are exactly two solutions to $\cos\theta = v$ if $-1 < v < 1$ and $0 \le \theta < 360°$. This fact is illustrated by considering the intersection of a vertical line with the unit circle. Any point x on the unit circle has coordinates $(\cos\theta, \sin\theta)$ where θ is the measure of the arc extending from $(1, 0)$ to x. Then the two points $(\cos\theta, \sin\theta)$ and $(\cos(-\theta), \sin(-\theta)) = (\cos\theta, -\sin\theta)$ share the x coordinate $\cos\theta$.

Next, note that

$$\cos(225°) = -\cos(225° - 180°) = -\cos(45°) = -\frac{\sqrt{2}}{2},$$

and that

$$\begin{aligned} \cos 105° &= \cos(60° + 45°) \\ &= \cos 60° \cos 45° - \sin 60° \sin 45° \\ &= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} - \sqrt{6}}{4}. \end{aligned}$$

Then $\cos(105°) + \cos(225°) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}-\sqrt{6}}{4} = -\frac{2\sqrt{2}}{4} + \frac{\sqrt{2}-\sqrt{6}}{4} = -\frac{\sqrt{2}+\sqrt{6}}{4}$.

Recognize this expression as $-\cos 15° = -\cos(-15) = \cos(-15 + 180) = \cos(165°)$. If not, note that $(-\frac{\sqrt{2}}{2})\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \cos(135°)\cos(30°) - \sin(135°)\sin(30°) = \cos(135° + 30°) = \cos(165°)$.

Finally, recall that $\cos(360° - \theta) = \cos\theta$. Then the other value for θ in the given range is $360° - 165° = 195°$, and the two solutions are $\boxed{165°, 195°}$.

As a final note, the simplest way to find the solution $\theta = 165°$ is to apply the sum of cosines formula: $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$. Then

$$\begin{aligned} \cos(225°) + \cos(105°) &= \cos(165° + 60°) + \cos(165° - 60°) \\ &= 2\cos(165°)\cos(60°) = 2\cos(165°)\frac{1}{2} = \cos(165°) \end{aligned}$$

Team Round

1. How many three digit positive integers s have the property that $s^2 - 1$ is a multiple of 3?

Solution: First, note that $s^2 - 1 = (s+1)(s-1)$. Therefore $s^2 - 1$ is a multiple of 3 whenever either $(s+1)$ is a multiple of 3 or $(s-1)$ is a multiple of 3.

Note that every third integer is a multiple of 3 $(3, 6, 9, 12, ...)$. Therefore among three consecutive integers $s-1, s$, and $s+1$, exactly one of the three is a multiple of 3.

If s is a multiple of 3, then neither $(s-1)$ nor $(s+1)$ are a multiple of 3, so $s^2 - 1$ is not a multiple of 3.

If s is not a multiple of 3, then either $(s-1)$ or $(s+1)$ must be a multiple of 3, and $s^2 - 1$ is a multiple of 3.

In summary, the three digit numbers s for which $s^2 - 1$ is a multiple of 3 are, in order,

$$100, 101, 103, 104, 106, 107, ..., 997, 998,$$

that is, all the three digit numbers not divisible by 3. Note that the condition is satisfied for two of every three consecutive three digit numbers, that there are 900 three digit numbers, and that 900 is divisible by three. Therefore there are $\frac{2}{3} \cdot 900 = \boxed{600}$ three digit numbers with the stated property.

The count of three digit numbers may be arrived at in multiple ways. For instance, there are 999 numbers from 1 to 999, and 99 of those, numbered from 1 to 99, are not three digit numbers, and $999 - 99 = 900$. Another way is to count the numbers in the sequence $100, 101, ..., 998, 999$.

2. Given: $\dfrac{a-b}{c-b} = -4$; find $\dfrac{a-c}{b-c}$.

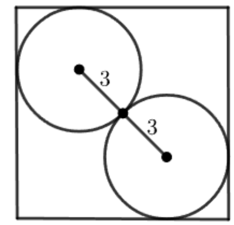
Solution: First, note that $\dfrac{a-b}{c-b} = -\dfrac{a-b}{b-c} = -4$, so that

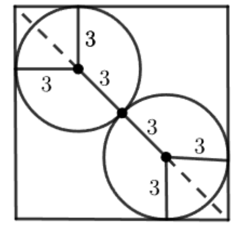
$$\frac{a-b}{b-c} = 4.$$

Next, note that:

$$\begin{aligned}\frac{a-c}{b-c} &= \frac{a-b+b-c}{b-c}\\ &= \frac{a-b}{b-c} + \frac{b-c}{b-c}\\ &= 4 + 1 = \boxed{5}.\end{aligned}$$

3. Two circles, externally tangent and each with radius 3, are inscribed in a square as shown at right. What is the perimeter of the square?





Solution: The perimeter of a square is four times the side length and the diagonal is $\sqrt{2}$ times the side length. Start by finding the length of the diagonal. One diagonal passes through the two circle centers as well as the point of tangency of the two circles. That diagonal consists of a segment from a corner of the square to a circle center, the segment connecting the centers of the circles, and another segment from the second circle center to the closest corner.
The circles are tangent to each other, so that the distance between their centers is the sum of their radii: $3 + 3 = 6$.
Next, draw the radii from each of the circle centers to the square at the two points of tangency, forming squares with side length 3. Then the diagonals of these smaller squares each have length $3\sqrt{2}$.
Now the diagonal length of the original square is the sum of these three segment lengths: $3\sqrt{2} + 6 + 3\sqrt{2} = 6 + 6\sqrt{2}$. Divide this length by $\sqrt{2}$ to find the side length of the original square: $\frac{6+6\sqrt{2}}{\sqrt{2}} = 3\sqrt{2} + 6$.

Finally, the perimeter of the square is $4 \cdot (6 + 3\sqrt{2}) = \boxed{24 + 12\sqrt{2}}$.

4. Solve the following equation for Real numbers x. Give your answer in the form $\log_b a$ where a and b are integers and b is as small as possible.

$$3^x - 4 - 5 \cdot 3^{-x} = 0$$

Solution: First, let $u = 3^x$. Note that u is strictly positive, that is, $u > 0$ because it is a real exponential function. Then solve the equation for u:

$$\begin{aligned} u - 4 - 5 \cdot u^{-1} &= 0 \\ u^2 - 4u - 5 &= 0 \\ (u - 5)(u + 1) &= 0 \end{aligned}$$

and $u = 5$ or $u = -1$. The second equation resulted after multiplying the top equation by u. There is no need to check the solution $u = 0$ because $u > 0$. For the same reason, discard the solution $u = -1$.

Now $u = 3^x = 5$. Therefore $x = \boxed{\log_3 5}$. Note that there are equivalent forms of x, for instance $\log_9 25$, but none with an integer base less than 3.

5. The period of a periodic function $f(x)$ is the smallest value T such that $f(x) = f(x-T)$ for all values of x. Find the period of $g(x) = \frac{3}{4}\cos\frac{2}{3}x$, where x is measured in radians. Express your answer in radians as a multiple of π.

Solution: The trigonometric function cosine (as well as sine, secant and cosecant) has a period of 2π. To see this, observe the unit circle: the values of the function repeat after completing a full circle, or 2π radians, and no smaller rotation will cause values to repeat for every value of x. Therefore $\cos(x - 2\pi) = \cos(x)$ for all x.

To find $g(x) = g(x - T)$ for some T, set $\frac{3}{4}\cos\frac{2}{3}x = \frac{3}{4}\cos(\frac{2}{3}(x - T))$, or more simply $\cos\frac{2}{3}x = \cos(\frac{2}{3}x - \frac{2}{3}T)$.

Comparing our expressions above, $2\pi = \frac{2}{3}T$ since the period of cosine is 2π. Therefore $T = \boxed{3\pi}$.

6. Lilly divides her favorite number by a smaller integer d resulting in a remainder of 24. She then divides twice her number by d resulting in a remainder of 11. Find d.

Solution: Because Lilly got a remainder of 24 when she divided her favorite number by d, she knows that her favorite number is equal to $d \cdot k + 24$ for some integer k.

Now twice that number will give a remainder of 11 when divided by d, so $2(d \cdot k + 24) = 2d \cdot k + 48$ will have a remainder of 11.

But as the $2d \cdot k$ part divides evenly by d, the 48 must have a remainder of 11 when divided by d.

This means that $48 = j \cdot d + 11$ for some integer j, and $j \cdot d = 48 - 11 = 37$. As 37 is prime, $j = 1$ and $d = \boxed{37}$.

7. Simplify the following expression:

$$(x+y)^3 - \left(x^3 + y^3\right)$$

Solution: First, expand $(x + y)^3$:

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) = (x^2+xy+xy+y^2)(x+y)\\ &= x^3 + x^2y + x^2y + xy^2 + x^2y + xy^2 + xy^2 + y^3\end{aligned}$$

Next, combine like terms: $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

Thus, $(x + y)^3 - (x^3 + y^3) = x^3 + 3x^2y + 3xy^2 + y^3 - x^3 - y^3 = \boxed{3x^2y + 3xy^2}$.

8. Given positive integers a and b, where $a^2 + b^2 = 23ab$, find the value of the following expression expressed as a single natural logarithm (ln):

$$\ln(a+b) - \frac{\ln a + \ln b}{2}$$

Solution: First, note that since $a^2 + b^2 = 23ab$, $(a+b)^2 = a^2 + 2ab + b^2 = 23ab + 2ab = 25ab$. Apply the identities $\sqrt{x}^2 = x$, $\sqrt{a} = a^{\frac{1}{2}}$ and $\log a^n = n \log a$:

$$\ln(a+b) = \ln(\sqrt{(a+b)^2}) = \frac{1}{2}\ln((a+b)^2) = \frac{1}{2}\ln(25ab)$$

Next, apply the identity $\ln x + \ln y = \ln xy$ to the other part of the expression:

$$\frac{\ln a + \ln b}{2} = \frac{\ln ab}{2} = \frac{1}{2}\ln ab$$

Substitute these expressions into the original expression and apply the identity $\ln x - \ln y = \ln \frac{x}{y}$:

$$\begin{aligned}
\ln(a+b) - \frac{\ln a + \ln b}{2} &= \frac{1}{2}\ln(25ab) - \frac{1}{2}\ln(ab) \\
&= \frac{1}{2}\left(\ln(25ab) - \ln(ab)\right) \\
&= \frac{1}{2}\ln\left(\frac{25ab}{ab}\right) \\
&= \frac{1}{2}\ln 25 = \ln\left(25^{\frac{1}{2}}\right) = \ln\sqrt{25} = \boxed{\ln 5}
\end{aligned}$$

9. Given functions $f(x) = \sin 3x$ and $g(x) = 2\sin x$, find all values of x (expressed in radians) where $0 \le x < 2\pi$, such that $f(x) = g(x)$.

Solution: First, recall the angle addition formula for $\sin x$: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Replace $3x$ with $2x + x$ and apply this formula: $\sin 3x = \sin 2x \cos x + \cos 2x \sin x$.

Next, apply the double angle formulas $\sin 2x = 2\sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$:

$$\begin{aligned}\sin 3x &= (2\sin x\cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x \\ &= 3\sin x\cos^2 x - \sin^3 x \\ &= \sin x\left(3\cos^2 x - \sin^2 x\right).\end{aligned}$$

Substitute this expression in the original equation: $2\sin x = \sin x\left(3\cos^2 x - \sin^2 x\right)$.

Thus, either $\sin x = 0$ or $2 = 3\cos^2 x - \sin^2 x$. In the first case, the two solutions for $0 \le x < 2\pi$ to $\sin x = 0$ are $x = 0$ and $x = \pi$.

In the second case, add $1 = \cos^2 x + \sin^2 x$: $3 = 4\cos^2 x$ Then $\cos^2 x = \frac{3}{4}$, and $\cos x = \pm\frac{\sqrt{3}}{2}$

There are four solutions to $\cos x = \pm\frac{\sqrt{3}}{2}$ in $0 \le x < 2\pi$. They are $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, $x = \frac{7\pi}{6}$, and $x = \frac{11\pi}{6}$.

In summary, $x \in \boxed{\left\{0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}}$.