Worcester County Mathematics League

Varsity Meet 3 - January 23, 2019

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League

Varsity Meet 3 - January 23, 2019

Answer Key

Round 1 - Similarity and Pythagorean Theorem

1. $\frac{75}{4}$ or $18\frac{3}{4}$ or 18.75
2. 12
3. 10

Round 2 - Algebra I

1. $x = 18$
2. $\frac{9}{4}$ or $2\frac{1}{4}$ or 2.25
3. 96

Round 3 - Functions

1. -3
2. $\frac{5}{8}$ or 0.625
3. 8

Round 4 - Combinatorics

1. 126
2. $n = 7$
3. $\binom{16}{2}, \binom{16}{14}, \binom{10}{3}, \binom{10}{7}$ (all four required)

Round 5 - Analytic Geometry

1. $k = 4$
2. $0x^2+1y^2-12x+6y+69 = 0$ (first five values required)
3. $(-1, 2)$

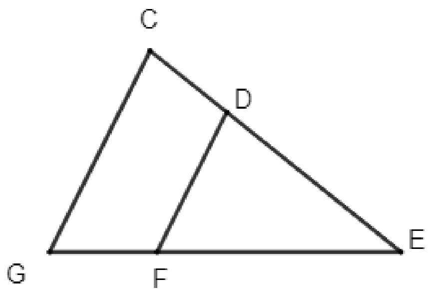
Team Round

1. $\frac{200\sqrt{3}}{9}$
2. $m = \frac{IR}{E - IR} = \frac{-IE}{IR - E}$
3. 26
4. 840
5. $6x + 4y + 3 = 0$
6. $x = -\frac{1}{4}$ or $\frac{1}{2}$, or $x = \left\{-\frac{1}{4}, \frac{1}{2}\right\}$
   (NB: "$-\frac{1}{4}$ and $\frac{1}{2}$" is not correct)
7. $\sqrt{\frac{50}{\pi}}$
8. $x = \frac{5}{2} = 2\frac{1}{2} = 2.5$ (only one solution)
9. 23

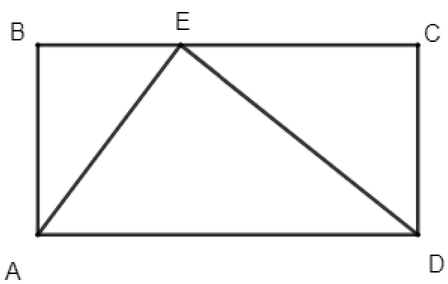
Worcester County Mathematics League
Varsity Meet 3 - January 23, 2019
Round 1 - Similarity and Pythagorean Theorem

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $\overline{DF} \parallel \overline{CG}$, $DF = 3$, $CG = 8$, and $CE = 30$, find CD.



2. Find the area of rectangle $ABCD$ if $AE = 3$, $DE = 4$, and $AD = 5$.



3. The medians of a right triangle that are drawn from the vertices of the acute angles have lengths $\sqrt{73}$ and $2\sqrt{13}$. Find the length of the hypotenuse.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Bromfield, Doherty, Tahanto

Worcester County Mathematics League
Varsity Meet 3 - January 23, 2019
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Solve for x: $x - 1 + 2x - 2 + 3x - 3 = 30 + 4x$

2. Simplify: $$\frac{x + y + \dfrac{x - 2y}{2}}{x + y - \dfrac{x + 3y}{3}}$$

3. The units digit of a two-digit number is 12 less than twice the tens digit. If the digits are reversed, the new number is 3 less than 8 times the tens digit of the original number. Find the original number.

ANSWERS

(1 pt) 1. $x =$ ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Southbridge, Clinton, Doherty

Worcester County Mathematics League
Varsity Meet 3 - January 23, 2019
Round 3 - Functions

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $f(x) = 2x + 5$ and $g(x) = 3x - 7$, find $f\left(g\left(f\left(-2\right)\right)\right)$.

2. Let rational function $m(x) = \dfrac{3x-2}{2x+1}$. Determine the value of $\left[m^{-1}\left(\dfrac{2}{3}\right)\right]^{-1}$.

3. Let $f(x)$ be a quadratic function where $f(-3) = f(1) = 0$ and $f(0) = -3$. If $g(x) = -2f(3-x)$, find the maximum value of $g(x)$.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Southbridge, Quaboag, QSC

Worcester County Mathematics League
Varsity Meet 3 - January 23, 2019
Round 4 - Combinatorics

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. An examination has nine questions from which the student is to select five to work on. How many distinct selections of five questions can be made?

2. If $\frac{{}_nP_6}{{}_nP_4} = 6$, find n.

3. Pascal's Triangle is a triangular array composed of binomial coefficients. Most coefficients appear twice in the Triangle, some appear three or four times, and one appears infinitely many times (the number 1). However, the first coefficient to appear exactly six or more times in the Triangle is the number 120. If two of the six instances are the result of calculating $\binom{120}{119}$ and $\binom{120}{1}$ and the other four instances appear within the first twenty rows of the triangle, find all four of those pairs of n and k such that $\binom{n}{k} = 120$.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. $n =$ ______________________

(3 pts) 3. $\binom{n}{k} = \left(\quad \right), \left(\quad \right), \left(\quad \right), \left(\quad \right)$

Bancroft, Algonquin, QSC

Worcester County Mathematics League
Varsity Meet 3 - January 23, 2019
Round 5 - Analytic Geometry

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find the value k for which the line $4x + ky = 5$ has equal x- and y-intercepts.

2. Find the equation for a parabola whose vertex is $(5, -3)$ and whose focus is $(8, -3)$. Express your answer in the form $Ax^2 + By^2 + Cx + Dy + E = 0$ where A, B, C, D, and E are all integers.

3. Three points lie on circle O: $A(2, -2)$, $B(3, 5)$, and $C(9, 5)$. Identify the point of intersection of line l_A and line l_B, two lines tangent to the circle at points A and B, respectively.

ANSWERS

(1 pt) 1. $k =$ ____________________

(2 pts) 2. _____ x^2+ _____ y^2+ _____ $x+$ _____ $y+$ _____ $= 0$ (fill in all five spaces)

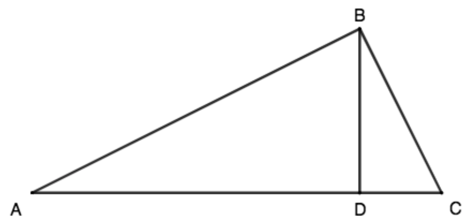
(3 pts) 3. (_____ , _____)

Shepherd Hill, Leicester, QSC

Worcester County Mathematics League
Varsity Meet 3 - January 23, 2019
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the area of right triangle ΔABC with altitude $\overline{BD}$, $\measuredangle A = 30°$, and $AD = 10$.



2. Express m in terms of I, E, and R:
$$I = \frac{mE}{R + mR}$$

3. Given that $f(x) = x^2 + 1$, find the sum of the coefficients of the polynomial $f(f(f(x)))$.

4. A committee of 3 students and 2 teachers is to be formed from a group of 8 students and 6 teachers. How many different committees can be formed?

5. Find the equation of the locus of all points in the Cartesian Plane that are equidistant from $3x + 2y = 7$ and $3x + 2y = -10$. Express your answer in the form $Ax + By + C = 0$ where A, B, and C are integers.

6. Solve for x:
$$4x + \frac{4}{1 + \frac{x}{1+x}} = 5$$

7. Water is poured at a constant rate into a cube whose edge length is 10 cm long. The water level rises to the halfway mark in the cube, at which point the water level starts to rise at half the original rate. The change occurs because inside the cube is a cylindrical can attached to the bottom of the cube. What is the radius of the can?

8. Solve for x:
$$\log_2(2x^2 + x - 3) - \log_2(3x^2 + x - 4) = 4 - \log_2 23$$

9. The continued fraction for the number $\frac{649}{200}$ can be expressed as $A + \cfrac{1}{B + \cfrac{1}{C + \cfrac{1}{D}}}$. Find $A + B + C + D$.

Worcester County Mathematics League
Varsity Meet 3 - January 23, 2019
Team Round Answer Sheet

ANSWERS

1. ____________________

2. $m =$ ____________________

3. ____________________

4. ____________________ committees

5. ____________________

6. $x =$ ____________________

7. ____________________

8. $x =$ ____________________

9. ____________________

Shrewsbury, St. John's, Tantasqua, Uxbridge, Quaboag, NDA, St. John's, Quaboag, QSC

Worcester County Mathematics League

Varsity Meet 3 - January 23, 2019

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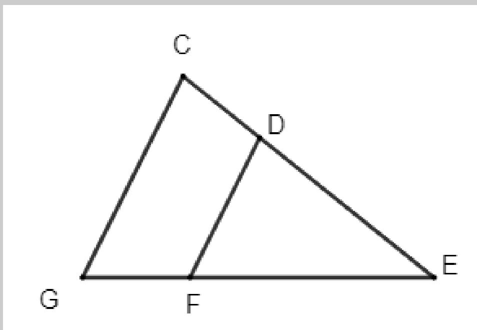
Round 1 - Similarity and Pythagorean Theorem

1. If $\overline{DF} \parallel \overline{CG}$, $DF = 3$, $CG = 8$, and $CE = 30$, find CD.

Solution:

Since $\overline{DF} \parallel \overline{CG}$, this means that two triangles are similar: $\Delta CGE \sim \Delta DFE$. From this we can set up a proportion with three of the four known quantities as well as an unknown quantity.



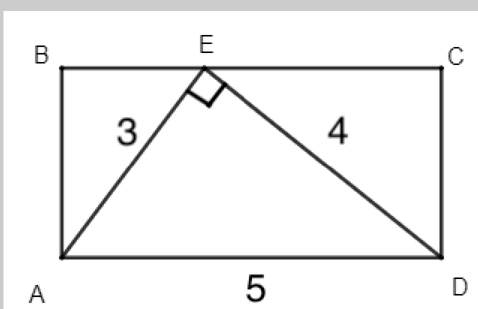
$$\frac{DF}{CG} = \frac{DE}{CE}$$
$$\frac{3}{8} = \frac{DE}{30}$$
$$\frac{90}{8} = DE$$
$$\frac{45}{4} = DE$$

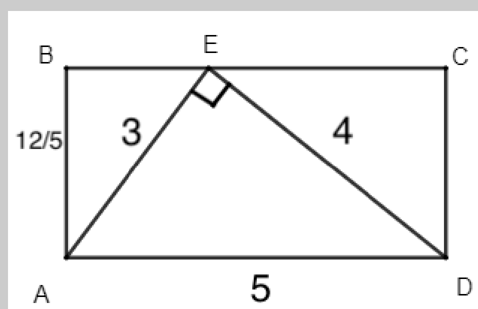
Since $CE - DE = CD$, then $30 - \frac{45}{4} = CD = \frac{120}{4} - \frac{45}{4} = \boxed{\frac{75}{4} = 18\frac{3}{4} = 18.75}$

2. Find the area of rectangle $ABCD$ if $AE = 3$, $DE = 4$, and $AD = 5$.

Solution:

Triangle ΔAED has lengths of 3, 4, and 5, which is a common Pythagorean Triple and indicate that the triangle is a right triangle.



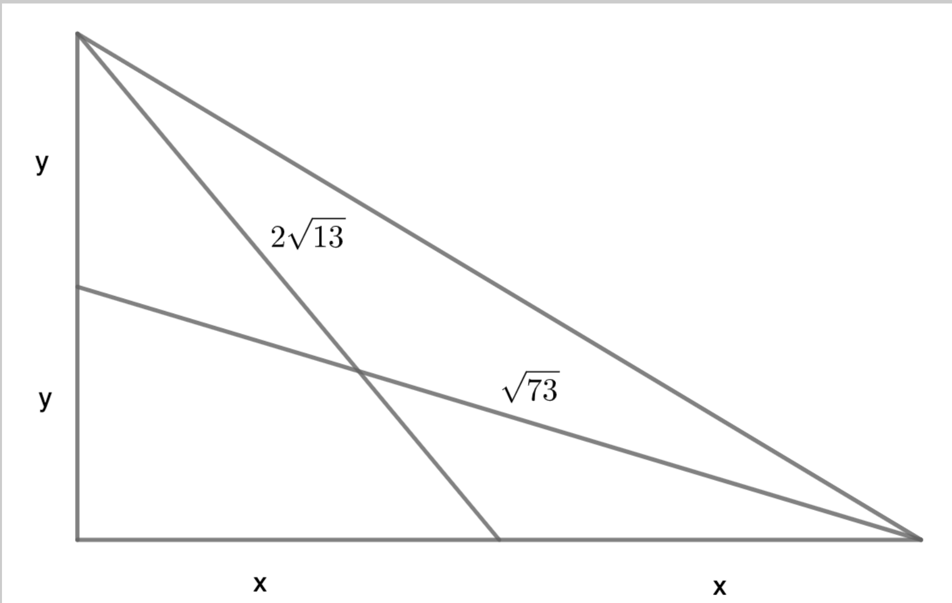


This triangle has area $\frac{1}{2} \cdot 3 \cdot 4 = 6$ since the legs can act as the base and height of the triangle, and because the area of a triangle is half the area of the associated rectangle, the area of rectangle $ABCD$ is $\boxed{12}$.

Alternate Solution: Triangle $\Delta AED \sim \Delta EBA$ with the lengths of δEBA being three-fifths of the lengths of ΔAED. We know the base of the rectangle (5) so we only need to know the height (AB). Noting that AB is three-fifths the length of DE, the height of the rectangle is $\frac{12}{5}$. The area of $ABCD$ is still $5 \cdot \frac{12}{5} = \boxed{12}$.

3. The medians of a right triangle that are drawn from the vertices of the acute angles have lengths $\sqrt{73}$ and $2\sqrt{13}$. Find the length of the hypotenuse.

Solution: Medians bisect the sides that they intersect, so we create and label a diagram.



This creates two additional right triangles and the following relationships.

$$(2x)^2 + y^2 = (\sqrt{73})^2 \qquad x^2 + (2y)^2 = (2\sqrt{13})^2$$
$$4x^2 + y^2 = 73 \qquad x^2 + 4y^2 = 52$$

Combining these equations:

$$\begin{array}{r} 4x^2 + y^2 = 73 \\ + \ (x^2 + 4y^2 = 52) \\ \hline 5x^2 + 5y^2 = 125 \end{array}$$

Our original goal is to find the length of the hypotenuse, or $\sqrt{(2x)^2 + (2y^2)}$.

$$5x^2 + 5y^2 = 125$$
$$x^2 + y^2 = 25$$
$$4x^2 + 4y^2 = 100$$
$$(2x)^2 + (2y^2) = 100$$
$$\sqrt{(2x)^2 + (2y)^2} = \boxed{10}$$

The system can also be solved by identifying the value of one of the variables, then the other, and then directly calculating the length of the hypotenuse.

Round 2 - Algebra I

1. Solve for x: $x-1+2x-2+3x-3=30+4x$

Solution: Combine like terms on each side and isolate x.

$$\begin{aligned} x-1+2x-2+3x-3 &= 30+4x \\ 6x-6 &= 30+4x \\ 2x &= 36 \\ x &= \boxed{18} \end{aligned}$$

2. Simplify: $\dfrac{x+y+\dfrac{x-2y}{2}}{x+y-\dfrac{x+3y}{3}}$

Solution: With this complex fraction, create a single fraction in the numerator and denominator by finding a common denominator for each expression.

$$\frac{x+y+\dfrac{x-2y}{2}}{x+y-\dfrac{x+3y}{3}} = \frac{\dfrac{2}{2}(x+y)+\dfrac{x-2y}{2}}{\dfrac{3}{3}(x+y)-\dfrac{x+3y}{3}} = \frac{\dfrac{2x+2y}{2}+\dfrac{x-2y}{2}}{\dfrac{3x+3y}{3}-\dfrac{x+3y}{3}} = \frac{\dfrac{(2x+2y)+(x-2y)}{2}}{\dfrac{(3x+3y)-(x+3y)}{3}}$$

From here, combine like terms, being careful to distribute appropriately when simplifying the denominator.

$$\frac{\dfrac{(2x+2y)+(x-2y)}{2}}{\dfrac{(3x+3y)-(x+3y)}{3}} = \frac{\dfrac{2x+2y+x-2y}{2}}{\dfrac{3x+3y-x-3y}{3}} = \frac{\dfrac{3x}{2}}{\dfrac{2x}{3}} = \frac{3x}{2}\cdot\frac{3}{2x} = \frac{9x}{4x} = \frac{9}{4}$$

The expression simplifies to $\boxed{\dfrac{9}{4} = 2\dfrac{1}{4} = 2.25}$.

3. The units digit of a two-digit number is 12 less than twice the tens digit. If the digits are reversed, the new number is 3 less than 8 times the tens digit of the original number. Find the original number.

Solution: Let the two-digit number be represented by the digits ab (not a product - a is the tens digit and b is the units digit/ones digit). The value of the number ab is equal to $10a+b$. If the units digit is 12 less than twice the tens digit, then we can set up an equation:

$$b = 2a-12$$

If the digits are reversed, and the new number (ba) is 3 less than 8 times the tens digit of the original number, we can set up another equation:

$$10b+a = 8a-3$$

We then solve the system to find a and b and determine the two-digit number. Substitute $2a - 12$ for b in the second equation:

$$\begin{aligned} 10(2a - 12) + a &= 8a - 3 \\ 20a - 120 + a &= 8a - 3 \\ 13a &= 117 \\ a &= 9 \end{aligned}$$

Since $a = 9$, $b = 2(9) - 12 = 6$ and the original number is $\boxed{96}$.

Round 3 - Functions

1. If $f(x) = 2x + 5$ and $g(x) = 3x - 7$, find $f(g(f(-2)))$.

Solution: The expression $f(g(f(-2)))$ is evaluated from the inside out. First, evaluate $f(-2)$:

$$f(-2) = 2(-2) + 5 = 1$$

Then replace $f(-2)$ with 1 to evaluate $f(g(1))$. Again, evaluate from the inside out. Evaluate $g(1)$:

$$g(1) = 3(1) - 7 = -4$$

Then replace $g(1)$ with -4. Finally, evaluate $f(-4)$:

$$f(-4) = 2(-4) + 5 = \boxed{-3}$$

2. Let rational function $m(x) = \dfrac{3x-2}{2x+1}$. Determine the value of $\left[m^{-1}\left(\dfrac{2}{3}\right)\right]^{-1}$.

Solution: Let $m^{-1}\left(\frac{2}{3}\right) = a$. Then $m(a) = \dfrac{2}{3}$. Plugging in a to the function m:

$$m(a) = \frac{3a-2}{2a+1} = \frac{2}{3}$$

Solving for a:

$$\begin{aligned}\frac{3a-2}{2a+1} &= \frac{2}{3}\\ 9a - 6 &= 4a + 2\\ 5a &= 8\\ a &= \frac{8}{5}\end{aligned}$$

Since $m^{-1}\left(\frac{2}{3}\right) = \dfrac{8}{5}$, it's reciprocal, $\left[m^{-1}\left(\frac{2}{3}\right)\right]^{-1} = \boxed{\dfrac{5}{8}}$.

Alternate solution: Find $m^{-1}(x)$ and plug in $\frac{2}{3}$.

$$\begin{aligned} m(x) &= \frac{3x-2}{2x+1} \\ y &= \frac{3x-2}{2x+1} \\ x &= \frac{3y-2}{2y+1} \\ 2yx + x &= 3y - 2 \\ 2yx - 3y &= -x - 2 \\ y(2x-3) &= -x-2 \\ y &= \frac{-x-2}{2x-3} \\ m^{-1}(x) &= \frac{-x-2}{2x-3} \end{aligned}$$

Evaluating $m^{-1}\left(\frac{2}{3}\right)$,

$$m^{-1}\left(\frac{2}{3}\right) = \frac{-\frac{2}{3}-2}{2\cdot\frac{2}{3}-3} = \frac{-\frac{2}{3}-\frac{6}{3}}{\frac{4}{3}-\frac{9}{3}} = \frac{\frac{-8}{3}}{\frac{-5}{3}} = \frac{-8}{-5} = \frac{8}{5}$$

and the reciprocal is $\boxed{\frac{5}{8}}$.

3. Let $f(x)$ be a quadratic function where $f(-3) = f(1) = 0$ and $f(0) = -3$. If $g(x) = -2f(3-x)$, find the maximum value of $g(x)$.

Solution: If f is a quadratic polynomial with x-intercepts at -3 and 1, then it can be expressed as $f(x) = a(x+3)(x-1)$ where $a \neq 0$. To determine the value of a, evaluate $f(0)$ knowing it must be equal to -3:

$$f(0) = a(0+3)(0-1) = -3a$$

If $-3a = -3$, then $a = 1$ and our function is $f(x) = (x+3)(x-1) = x^2 + 2x - 3 = (x+1)^2 - 4$.

The function $g(x) = -2f(3-x)$. We can either analyze this as a transformation or as a composite function.

As a transformation: Since $f(x) = (x+1)^2 - 4$, it represents an upward-facing parabola with vertex $(-1,-4)$. $g(x)$, with a claimed maximum, must be an downward-facing parabola with the maximum occuring at the vertex. Analyzing the transformation:

$$f(x) \Rightarrow f(-x) \text{ (reflect over } y\text{-axis)}$$

$$f(-x) \Rightarrow f(-x+3) = f(-(x-3)) = f(3-x) \text{ (shift function right 3 units)}$$

$$f(3-x) \Rightarrow -2f(3-x) \text{ (reflect over } x\text{-axis and stretch vertically by a factor of 2)}$$

Applying these transformations to the point $(-1,-4)$, the vertex of f we find

$$(-1,-4) \Rightarrow (1,-4) \Rightarrow (4,-4) \Rightarrow (4,8)$$

and the vertex of $g(x)$ will be at $(4,8)$. The maximum of this function is $\boxed{8}$.

As a composite function: Since $f(x) = (x+3)(x-1) = x^2 + 2x - 3 = (x+1)^2 - 4$, we can find $g(x) = -2f(3-x)$ by evaluating f at $3-x$ and then multiplying that by -2. There's only one x in the vertex form of f, so let's use that.

$$\begin{aligned} g(x) = -2f(3-x) &= -2\left[((3-x)+1)^2 - 4\right] \\ &= -2\left[(4-x)^2 - 4\right] \\ &= -2\left[16 - 8x + x^2 - 4\right] \\ &= -2(x^2 - 8x + 12) \\ &= -2x^2 + 16x - 24 \end{aligned}$$

The vertex of this downward-facing parabola will represent the maximum. Find the x-coordinate:

$$x = \frac{-b}{2a} = \frac{-16}{2(-2)} = 4$$

And then find the y-coordinate:

$$g(4) = -2(16) + 16(4) - 24 = -32 + 64 - 24 = \boxed{8}.$$

Round 4 - Combinatorics

1. An examination has nine questions from which the student is to select five to work on. How many distinct selections of five questions can be made?

Solution: Since the order of the questions does not matter in this case, we can evaluate ${}_9C_5$ which is the number of combinations of 5 objects that can be chosen from 9 objects.

$$ {}_9C_5 = \frac{9!}{(9-5)!5!} = \frac{9!}{4!5!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} $$

Then we simplify.

$$ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2} = 9\cdot 7\cdot 2 = \boxed{126} $$

2. If $\dfrac{{}_nP_6}{{}_nP_4} = 6$, find n.

Solution: The value of ${}_nP_6$ is $\frac{n!}{(n-6)!}$ and the value of ${}_nP_4$ is $\frac{n!}{(n-4)!}$. Their quotient is

$$ \frac{{}_nP_{6!}}{{}_nP_4} = \frac{\frac{n!}{(n-6)!}}{\frac{n!}{(n-4)!}} = \frac{n!(n-4)!}{n!(n-6)!} = \frac{(n-4)(n-5)(n-6)(n-7)\cdots 3\cdot 2\cdot 1}{(n-6)(n-7)\cdots 3\cdot 2\cdot 1} = (n-4)(n-5) $$

Setting this equal to 6:

$$ \begin{aligned} (n-4)(n-5) &= 6 \\ n^2-9n+20 &= 6 \\ n^2-9n+14 &= 0 \\ (n-7)(n-2) &= 0 \end{aligned} $$

So $n=7$ or $n=2$. Since ${}_2P_6$ and ${}_2P_4$ make no sense, $\boxed{n=7}$.

3. Pascal's Triangle is a triangular array composed of binomial coefficients. Most coefficients appear twice in the Triangle, some appear three or four times, and one appears infinitely many times (the number 1). However, the first coefficient to appear exactly six or more times in the Triangle is the number 120. If two of the six instances are the result of calculating $\binom{120}{119}$ and $\binom{120}{1}$ and the other four instances appear within the first twenty rows of the triangle, find all four of those pairs of n and k such that $\binom{n}{k} = 120$.

Solution: There's no direct way to determine which combination of n and k results in $\binom{n}{k} = \frac{n!}{(n-k)!k!} = 120$, but we can use some intuition to narrow down the possibilities. First of all, Pascal's Triangle begins with:

$$
\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

It looks like it will take a little time to reach 120 (it will, but not too long) and we can eliminate the values of 1, 2, 3, and 4 from being n. We don't ever need to check $\binom{n}{0} = \binom{n}{n}$ since that is equal to 1, nor $\binom{n}{1} = \binom{n}{n-1}$ since that is equal to n, and neither of these values will be 120 for n less than or equal to 20. Additionally, due to the symmetry of Pascal's Triangle, we only need to check approximately half of the entries. Finally, since $5! = 120$ as a permutation, the combination will be less than that, so let's start the search from $n = 6$. We know $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, so let's see what happens when we evaluate $\binom{6}{4}$:

$$\binom{6}{4} = \frac{6!}{2!4!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot4\cdot3\cdot2\cdot1} = \frac{6\cdot5\cdot\cancel{(4\cdot3\cdot2\cdot1)}}{2\cdot1\cdot\cancel{(4\cdot3\cdot2\cdot1)}}$$

Notice that all the factors of 6! that are also factors of 4! are cancelled out immediately, with only 6 and 5 remaining to be divided by 2 and 1, giving us a value of 15. So what factors do we want remaining in our calculation to give us a value of 120? Well, $120 = 2^3 \cdot 3 \cdot 5$, so the only prime factors that can be left over in the numerator once we cancel $k!$ from $n!$ and then cancel $(n-k)!$ from the remainder are three 2s, one 3, and one 5. With $n = 6$, this is impossible, since $\binom{6}{2} = \binom{6}{4} = 15$ and $\binom{6}{3} = \frac{6!}{3!3!} = 20$. Move onto $n = 7$.

If $n = 7$, let's try $\binom{7}{5} = \frac{7!}{5!2!} = \frac{7\cdot6}{2\cdot1} = 21$. This isn't desirable for two reasons, one being that 21 isn't 120 and the other being that 21 has a factor of 7 and we can't have that for 120. To avoid 7 being a factor in $\binom{7}{k}$ we'd need to do $\binom{7}{7}$ or $\binom{7}{0}$, and both of those are equal to 1, so $n = 7$ is also out.

If $n = 8$ we still need to cancel 7, but choosing $\binom{8}{7} = \binom{8}{1}$ so $n = 8$ is out. What about $n = 9$? We'd need to choose $\binom{9}{7} = \frac{9\cdot8}{2\cdot1} = 36$. Also out. What about $n = 10$? To cancel 7, we'd need $\binom{10}{7} = \frac{10\cdot9\cdot8}{3\cdot2\cdot1} = 5\cdot3\cdot8 = 120$(!!!). Since $\binom{10}{7} = \binom{10}{3}$, these are two of our four answers.

What about $n = 11$? Can't, since 11 isn't a factor of 120. $n = 12$ will give us, at best, $\binom{12}{11} = 12$ since we need to cancel out the 11. $n = 13$ is out because 13 isn't a factor of 120, and $n = 14$ is out because 7 is a factor of 14 and 7 isn't a factor of 120. $n = 15$ would require, at best, $\binom{15}{14} = 15$ so that's out. And $n = 16$ would require, at best, $\binom{16}{14}$ to get rid of the factor of 7. But wait! $\binom{16}{14} = \frac{16\cdot15}{2\cdot1} = 8\cdot15 = 120$(!!!). So there we have it: $\boxed{120 = \binom{10}{7} = \binom{10}{3} = \binom{16}{14} = \binom{16}{2}}$

Round 5 - Analytic Geometry

1. Find the value k for which the line $4x + ky = 5$ has equal x- and y-intercepts.

Solution: To find intercepts, plug in 0 for one variable and solve for the other. We do this twice. To find the x-intercept, set $y = 0$:

$$4x + ky = 5 \Rightarrow 4x + k(0) = 5 \Rightarrow x = \frac{5}{4}$$

To find the y-intercept, set $x = 0$:

$$4x + ky = 5 \Rightarrow 4(0) + ky = 5 \Rightarrow y = \frac{5}{k}$$

If $\frac{5}{4} = \frac{5}{k}$, then $\boxed{k = 4}$.

2. Find the equation for a parabola whose vertex is $(5, -3)$ and whose focus is $(8, -3)$. Express your answer in the form $Ax^2 + By^2 + Cx + Dy + E = 0$ where A, B, C, D, and E are all integers.

Solution: For a vertex of $(5, -3)$ and focus of $(8, -3)$, this is a rightward-facing parabola, meaning it takes the form of

$$(y - k)^2 = 4p(x - h)^2$$

where (h, k) is the vertex of the parabola and p is the distance from the vertex to the focus. This means our equation is

$$(y + 3)^2 = 12(x - 5)$$

and expanding it gives us

$$\begin{aligned} (y + 3)^2 &= 12(x - 5) \\ y^2 + 6y + 9 &= 12x - 60 \\ y^2 - 12x + 6y + 69 &= 0 \end{aligned}$$

As per the answer guidelines, the equation is $\boxed{0x^2 + 1y^2 - 12x + 6y + 69 = 0}$.

3. Three points lie on circle O: $A(2, -2)$, $B(3, 5)$, and $C(9, 5)$. Identify the point of intersection of line l_A and line l_B, two lines tangent to the circle at points A and B, respectively.

Solution: There are a few ways to determine the equation of a circle from three points. One is to find the intersection of the perpendicular bisectors to two chords of the circle, which will be at the circle's center. Another is to leverage the fact that all three points are equidistant from the center and set up a system of equations. In this particular case, we are lucky - one of the chords is horizontal, meaning the perpendicular bisector is vertical, and therefore we can quickly find the x-coordinate of the center.

Consider the chord $\overline{BC}$ which has endpoints of $B(3,5)$ and $C(9,5)$. The midpoint is at $(6,5)$ and the line perpendicular to the chord is $x = 6$. Knowing that the x-coordinate of the center is 6, we can now find the point equidistant from A and B with x-coordinate 6. Using the distance formula:

$$\sqrt{(x-2)^2 + (y+2)^2} = \sqrt{(x-3)^2 + (y-5)^2}$$

$$(6-2)^2 + (y+2)^2 = (6-3)^2 + (y-5)^2$$

$$16 + y^2 + 4y + 4 = 9 + y^2 - 10y + 25$$

$$20 + 4y = 34 - 10y$$

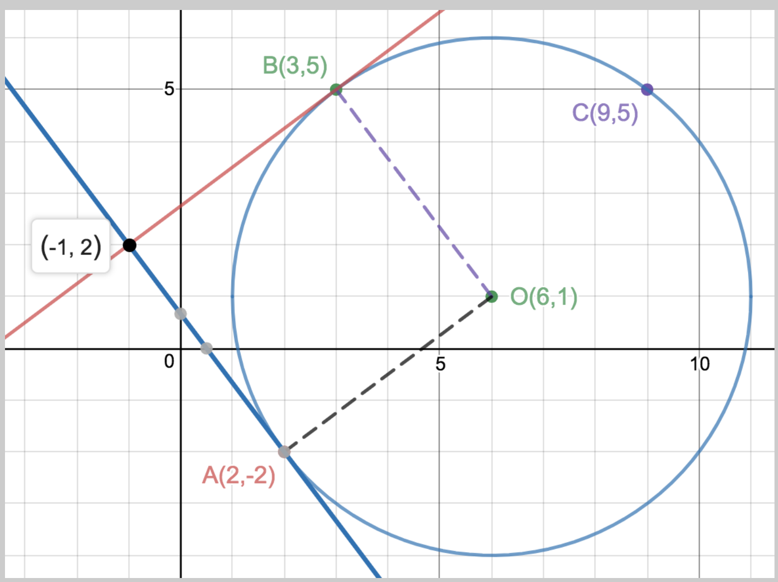
$$14y = 14$$

$$y = 1$$

The center is at $(6,1)$. The radius is

$$\sqrt{(9-6)^2 + (5-1)^2} = \sqrt{9+16} = 5$$

and the circle's equation is $(x-6)^2 + (y-1)^2 = 25$. Let the center of the circle be $O(6,1)$. Our goal is to find the intersection of two of the circle's tangent lines, one at $A(2,-2)$ and one at $B(3,5)$.



Tangent lines to circles have a slope that is the opposite reciprocal of the line connecting the point of tangency to the center. We know the lines tangent to $A(2,-2)$ and $B(3,5)$ have equations of

$$y + 2 = [\text{opposite reciprocal of slope of } \overline{AO}](x-2)$$

$$y - 5 = [\text{opposite reciprocal of slope of } \overline{BO}](x-3)$$

The slope of $\overline{AO}$ is $\frac{1-(-2)}{6-2} = \frac{3}{4}$ and the slope of $\overline{BO}$ is $\frac{1-5}{6-3} = -\frac{4}{3}$. Plugging in the appropriate values our lines are now:

$$y + 2 = -\frac{4}{3}(x - 2)$$

$$y - 5 = \frac{3}{4}(x - 3)$$

Solving for y in each and the substituting to find x:

$$-\frac{4}{3}(x - 2) - 2 = \frac{3}{4}(x - 3) + 5$$

$$-16(x - 2) - 24 = 9(x - 3) + 60$$

$$-16x + 32 - 24 = 9x - 27 + 60$$

$$-25 = 25x$$

$$-1 = x$$

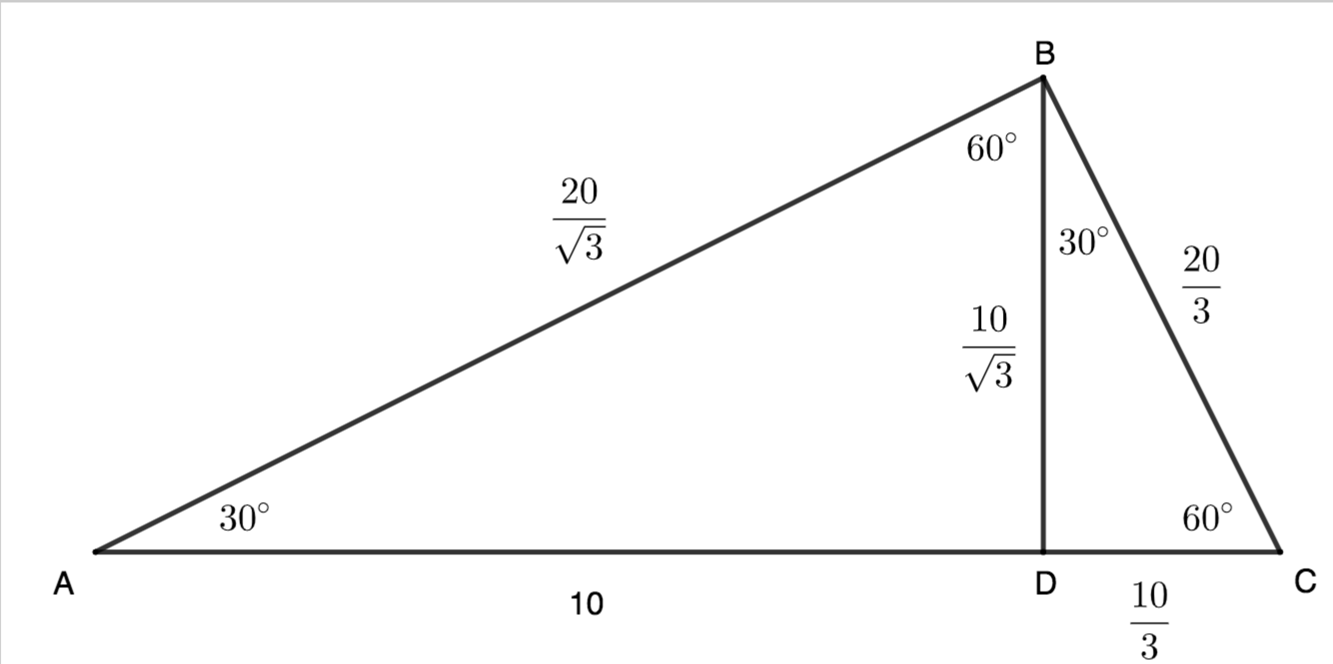
With $x = -1$, $y = \frac{3}{4}(-1 - 3) + 5 = \frac{3}{4}(-4) + 5 = -3 + 5 = 2$ for an intersection point of $\boxed{(-1, 2)}$.

Alternate approach: Note that the radii $\overline{AO}$ and $\overline{BO}$ are perpendicular and each has length 5. This means that the point of intersection that we'll call P will be on square $AOBP$ with side length 5 and be opposite point O. Using 3-4-5 right triangles, draw the square and identify the intersection point.

Team Round

1. Find the area of right triangle ΔABC with altitude $\overline{BD}$, $\measuredangle A = 30°$, and $AD = 10$.

Solution: Since $\overline{BD}$ is an altitude, it makes a right angle with $\overline{AC}$ and gives us two smaller $30° - 60° - 90°$ right triangles. Using the information given, we can fill out each length.



Since the area of a right triange is half the product of the legs, we arrive at

$$\frac{1}{2}\frac{20}{\sqrt{3}} \cdot \frac{20}{3} = \frac{200}{3\sqrt{3}} = \frac{200\sqrt{3}}{9}$$

Alternatively, we could look at this from a traditional base/height perspective and find the area to (also) be

$$\frac{1}{2}\left(10 + \frac{10}{3}\right) \cdot \frac{10}{\sqrt{3}} = \frac{40}{3} \cdot \frac{5}{\sqrt{3}} = \frac{200}{3\sqrt{3}} = \frac{200\sqrt{3}}{9}$$

You could also find the area of each of the smaller triangles, but that's essentially the same thing as what we just did:

$$\left(\frac{1}{2} \cdot \frac{10}{\sqrt{3}} \cdot 10\right) + \left(\frac{1}{2} \cdot \frac{10}{\sqrt{3}} \cdot \frac{10}{3}\right) = \frac{1}{2} \cdot \frac{10}{\sqrt{3}} \cdot \left(10 + \frac{10}{3}\right)$$

Either way, the area is $\boxed{\frac{200\sqrt{3}}{9}}$.

2. Express m in terms of I, E, and R:

$$I = \frac{mE}{R + mR}$$

Solution: Isolate m by multiplying both sides by the value of the denominator, separating terms involving m from terms not involving m, factoring out m, and then dividing both sides of the equation by the other factor.

$$\begin{aligned} I &= \frac{mE}{R + mR} \\ I(R + mR) &= mE \\ IR + IRm &= mE \\ IRm - mE &= -IR \\ m(IR - E) &= -IR \\ \boxed{m = \frac{-IR}{IR - E} = \frac{IR}{E - IR}} \end{aligned}$$

3. Given that $f(x) = x^2 + 1$, find the sum of the coefficients of the polynomial $f(f(f(x)))$.

Solution: Since

$$\begin{aligned} f(f(f(x))) &= f(f(x^2 + 1)) \\ &= f((x^2 + 1)^2 + 1) \\ &= f(x^4 + 2x^2 + 2) \\ &= (x^4 + 2x^2 + 2)^2 + 1 \\ &= x^8 + 2x^6 + 2x^4 + 2x^6 + 4x^4 + 4x^2 + 2x^4 + 4x^2 + 4 + 1 \\ &= x^8 + 4x^6 + 8x^4 + 8x^2 + 5 \end{aligned}$$

The sum of the coefficients is $1 + 4 + 8 + 8 + 5 = \boxed{26}$.

Alternate method: Plugging in 1 to a polynomial function/expression results in the sum of the coefficients. Therefore, evaluate $f(f(f(1)))$.

$$f(f(f(1))) = f(f(2)) = f(5) = \boxed{26}$$

4. A committee of 3 students and 2 teachers is to be formed from a group of 8 students and 6 teachers. How many different committees can be formed?

Solution: The total number of committees that can be formed is the product of the number of possible student subcommittees by the number of possible teacher subcommittees, or the product of $nCr83$ and ${}_6C_2$.

$${}_8C_3 \cdot {}_6C_2 = \frac{8!}{5!3!}\frac{6!}{4!2!} = \frac{8 \cdot 7 \cdot 6 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 3 \cdot 5 = \boxed{840 \text{ possible committees}}$$

5. Find the equation of the locus of all points in the Cartesian Plane that are equidistant from $3x + 2y = 7$ and $3x + 2y = -10$. Express your answer in the form $Ax + By + C = 0$ where A, B, and C are integers.

Solution: The locus must be a line itself since it is equidistant between two lines (and also because the answer must be in the form of a line). The line must be parallel to each of the other two lines, so we rewrite the lines in slope-intercept form:

$$\begin{aligned} 3x + 2y &= 7 \\ 2y &= -3x + 7 \\ y &= -\frac{3}{2}x + \frac{7}{2} \end{aligned} \qquad \begin{aligned} 3x + 2y &= -10 \\ 2y &= -3x - 10 \\ y &= -\frac{3}{2}x - 5 \end{aligned}$$

Each of these lines has slope $\frac{3}{2}$, except one has y-intercept $\frac{7}{2}$ and the other has y-intercept -5. The line between these two should have y-intercept $\frac{1}{2}\left(\frac{7}{2} - 5\right) = -\frac{3}{4}$. We can take this to be the y-intercept of our equidistant line.

$$\begin{aligned} y &= \frac{3}{2}x - \frac{3}{4} \\ 4y &= -6x - 3 \\ &\boxed{6x + 4y + 3 = 0} \end{aligned}$$

Explanation of averaging intercepts: The appropriate method for finding the answer to this problem would seem to be

- Find slopes of lines.
- Find point on one line, and perpendicular line through that point.
- Find intersection of perpendicular line with other line.
- Find midpoint between those two points.
- Create solution line with parallel slope through midpoint.

This is a lot of work with the given information. So we consider an easier approach.

We are given two parallel lines l_1 and l_2. The solution line we will call l_3. We know all three lines have the same slope and have three different y-intercepts. If we adjust the slope of all three of these lines simultaneously while keeping their y-intercepts the same, then we will always have line l_3 equidistant from l_1 and l_2. If this is the case, let the slope be zero and l_1 and l_2 are $y = \frac{7}{2}$ and $y = -10$, horizontal lines where the line l_3, which is immediately between them, must have equation $y = -\frac{3}{4}$. Readjust the slopes back to $-\frac{3}{2}$ and proceed from there.

6. Solve for x:

$$4x + \frac{4}{1 + \frac{x}{1+x}} = 5$$

Solution: Begin by simplifying the complex fraction.

$$\frac{4}{1+\frac{x}{1+x}} = \frac{4}{\frac{1+x}{1+x}+\frac{x}{1+x}} = \frac{4}{\frac{2x+1}{x+1}} = \frac{4x+4}{2x+1}$$

Substituting back in to the original equation:

$$\begin{aligned} 4x + \frac{4x+4}{2x+1} &= 5 \\ 8x^2 + 4x + 4x + 4 &= 10x + 5 \\ 8x^2 - 2x - 1 &= 0 \\ (4x+1)(2x-1) &= 0 \end{aligned}$$

There are two solutions for x: $\boxed{-\frac{1}{4} \text{ and } \frac{1}{2}}$.

7. Water is poured at a constant rate into a cube whose edge length is 10 cm long. The water level rises to the halfway mark in the cube, at which point the water level starts to rise at half the original rate. The change occurs because inside the cube is a cylindrical can attached to the bottom of the cube. What is the radius of the can?

Solution: The cube has volume 1000 cm^3. As the water is being poured into the cube at a constant rate, we would expect the water level to rise at a constant rate as well. However, we know that the can at the bottom is taking up some volume that the water doesn't need to fill, but how big is the can?

If, at the halfway point of the cube, the water level rises at half the rate, it must be filling up twice the volume. This means that the can takes up half the volume in the lower half of the cube, which means the can has volume 250 cm^3. It also must have height 5 cm since it ends halfway up the cube.

The volume of a cylinder is $V = \pi r^2 h$. We know the height and volume of the can, so we solve for the radius.

$$\begin{aligned} 250 &= \pi r^2(5) \\ \frac{50}{\pi} &= r^2 \\ \boxed{\sqrt{\frac{50}{\pi}} = r} \end{aligned}$$

8. Solve for x:

$$\log_2(2x^2+x-3) - \log_2(3x^2+x-4) = 4 - \log_2 23$$

Solution:

$$\begin{aligned}
\log_2(2x^2+x-3) - \log_2(3x^2+x-4) &= 4 - \log_2 23 \\
\log_2(2x^2+x-3) - \log_2(3x^2+x-4) &= \log_2 16 - \log_2 23 \\
\log_2\left(\frac{2x^2+x-3}{3x^2+x-4}\right) &= \log_2\left(\frac{16}{23}\right) \\
\frac{2x^2+x-3}{3x^2+x-4} &= \frac{16}{23} \\
46x^2+23x-69 &= 48x^2+16x-64 \\
0 &= 2x^2-7x+5 \\
0 &= (2x-5)(x-1)
\end{aligned}$$

Our answers are $\frac{5}{2}$ and 1 but plugging in $x=1$ gives $\log_2(2+1-3) = \log_2 0$ which is undefined. Our only solution is $\boxed{x = \frac{5}{2} = 2\frac{1}{2} = 2.5}$.

9. The continued fraction for the number $\dfrac{649}{200}$ can be expressed as $A + \cfrac{1}{B + \cfrac{1}{C + \cfrac{1}{D}}}$. Find $A+B+C+D$.

Solution: The approach we take is iterative. Determine the number as a mixed fraction and remove the integer part. Reciprocate the rational part and repeat the process until you are left with an integer as the reciprocal.

$$\begin{aligned}
\frac{649}{200} &= 3\frac{49}{200} \Rightarrow \text{A is 3} \\
\frac{49}{200} \Rightarrow \frac{200}{49} &= 4\frac{4}{49} \Rightarrow \text{B is 4} \\
\frac{4}{49} \Rightarrow \frac{49}{4} &= 12\frac{1}{4} \Rightarrow \text{C is 12} \\
\frac{1}{4} \Rightarrow \frac{4}{1} &= 4 \Rightarrow \text{D is 4}
\end{aligned}$$

We represent $\dfrac{649}{200} = 3 + \cfrac{1}{4 + \cfrac{1}{12 + \cfrac{1}{4}}} = A + \cfrac{1}{B + \cfrac{1}{C + \cfrac{1}{D}}}$ so $A+B+C+D = \boxed{23}$.