WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014

Round 1: Evaluation of Algebraic Expressions and Order of Operations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Evaluate: $16 \div \left(-\frac{2}{3}+2\right)\left(\frac{1}{4}\right)+5$

2. Evaluate the following expression when $x=-\frac{3}{2}$

$$4(x+1)^3-2(x+2)^2-8(x+1)+6$$

3. If $x=3t^2+2t+1$, evaluate $\sqrt{x(2t^2-1)}$ when $t=3$.

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________

(3 pts.) 3. ______________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014
Round 2: Solving Linear Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve for x in the expression $\frac{1}{3}x + 17 = 3x - 7$

2. A number is cut in half and is then added to $\frac{8}{3}$. This sum is equal to the difference of $\frac{3}{5}$ of the number and $\frac{9}{4}$. What is the number?

3. Find the solution(s) to the following equation:

$|x|+|x+6|=10$

ANSWERS

(1 pt.) 1. ________________________________

(2 pts.) 2. ________________________________

(3 pts.) 3. ________________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014
Round 3: Logic Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. What is the sum of all the integers from -17 to +20?

2. Several people are in a line. Starting from one end Jane is counted as the 4th person in line and starting from the other end she is counted as the 13th person in line. How many total people are in the line?

3. Old MacDonald had a farm, and he also had a love of logic puzzles. Using a code where each letter represents a different single digit, he noticed that COW = 306 and that

$$\begin{array}{r} \text{GOOSE} \\ -\ \underline{\text{DUCK}} \\ \text{EGG} \end{array}$$

What is the four-digit number represented by the word DUCK?

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________ people in line

(3 pts.) 3. DUCK = ____________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014
Round 4: Ratios, Proportions, and Variation

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If 8 is 40% of x and 12 is 80% of y, then y is what percentage of x?

2. Suppose that c is directly proportional to the cube of d and inversely proportional to the square of e. If c = 32 when d = 4 and e = 6, find the value of c when d = 6 and e = 12.

3. There are 2820 tractors in Farmtown. The ratio of the number of medium tractors to the number of small tractors is 7:5 and the ratio of the number of small tractors to the number of large tractors is 8:9. How many large tractors are there in Farmtown?

ANSWERS

(1 pt.) 1. ________________________________%

(2 pts.) 2. ________________________________

(3 pts.) 3. ________________________________ large tractors

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

APPROVED CALCULATORS ALLOWED

1. If distinct beads A,B,C,D and E are strung on a circular bracelet, ABCDE, BCDEA and EDCBA are all orderings for the same stringing. How many different stringings can be made using these five beads?

2. The perimeter of a rectangle is 20 units, and the length of the rectangle is 1.75 times the width. If we increase the perimeter of the rectangle to 30 units while keeping the length-to-width proportion constant, by what percentage does the area of the rectangle increase?

3. If $a \diamond b = a^b$ and $a \star b = ab^2$, evaluate $128 \diamond \dfrac{(16 \star 2) \diamond \frac{1}{3}}{7 \star \left(2048 \diamond \frac{1}{11}\right)}$

4. Suppose that the letters A,B,C,D and E each represent a distinct digit. Determine the value of ABCDE if

$$\begin{array}{r} \text{A B C D E} \\ \times \quad\quad\;\; 4 \\ \hline \text{E D C B A} \end{array}$$

5. 17 apples cost as much as 6 cherries. 18 cherries cost as much as 34 bunches of grapes. Kate spent twice as much money on grapes as she did on apples. If she bought 8 bunches of grapes and no cherries, how many apples did Kate buy?

6. Solve for y in terms of x and simplify: $2y + 2x = 3(4 - 6yx) - 4(-x) - 12y$

7. Michael is walking from his home to his school. At 8:10, he has walked one fourth of the total distance. At 8:12, he has walked one third of the total distance. Assuming Michael walks at a constant rate of speed, at what time did Michael leave his home?

8. Jake took an exam that had 200 multiple choice questions. Each correct answer earned him 4 points and each incorrect answer deducted 2 points from his score. If Jake answered every question on the exam and he received a total score of 536, how many questions did he answer incorrectly?

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014
Team Round Answer Sheet

1. ______________________________stringings

2. ______________________________%

3. ______________________________

4. ABCDE = ___________________

5. ______________________________ apples

6. y = _________________________

7. ______________________________

8. ______________________________ incorrect answers

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014 ANSWER KEY

Round 1:

1. 8 (St. John's)
2. 9 (West Boylston)
3. $17\sqrt{2}$ (AMSA)

Round 2:

1. 9 (Bartlett)
2. $\frac{295}{6}$ or $49\frac{1}{6}$ (Douglas)
3. -8 and 2 (Auburn)

Round 3:

1. 57 (South)
2. 16 (St. John's)
3. DUCK=9237 (Notre Dame Academy)

Round 4:

1. 75% (Worcester Academy)
2. 27 (Hopedale)
3. 900 (West Boylston)

TEAM Round

1. 12
2. 125% (Notre Dame Academy)
3. 2 (South)
4. ABCDE = 21978 (Bartlett)
5. 6 (St. John's)
6. $y = \frac{6+x}{7+9x}$ (St. John's)
7. 8:04 (Worcester Academy)
8. 44 (Douglas)

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 1 - October 29, 2014 SOLUTIONS

Round 1: Evaluation of Algebraic Expressions and Order of Operations

1. Evaluate: $16 \div \left(-\frac{2}{3}+2\right)\left(\frac{1}{4}\right)+5$

SOLUTION:

$$16 \div \left(-\frac{2}{3}+2\right)\left(\frac{1}{4}\right)+5$$
$$16 \div \left(\frac{4}{3}\right)\left(\frac{1}{4}\right)+5$$
$$16\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)+5$$
$$4 \times \frac{3}{1} \times \frac{1}{4}+5$$
$$3+5=8$$

2. Evaluate the following expression when $x=-\frac{3}{2}$

$$4(x+1)^3-2(x+2)^2-8(x+1)+6$$

SOLUTION:

$$4(x+1)^3-2(x+2)^2-8(x+1)+6$$
$$4\left(-\frac{1}{2}\right)^3-2\left(\frac{1}{2}\right)^2-8\left(-\frac{1}{2}\right)+6$$
$$-\frac{4}{8}-\frac{2}{4}+\frac{8}{2}+6$$
$$-\frac{1}{2}-\frac{1}{2}+4+6=9$$

3. If $x=3t^2+2t+1$, evaluate $\sqrt{x(2t^2-1)}$ when $t=3$.

SOLUTION:

$$x=3(3)^2+2(3)+1$$
$$x=27+6+1$$
$$x=34$$

Now substitute x into the expression:

$$\sqrt{34(2(3)^2-1)}$$
$$=\sqrt{34(18-1)}$$
$$=\sqrt{34(17)}$$
$$=\sqrt{17^2(2)}$$
$$=17\sqrt{2}$$

Round 2: Solving Linear Equations

1. Solve for x in the expression $\frac{1}{3}x+17=3x-7$

SOLUTION:

Begin by multiplying both sides by 3:

$$x+51=9x-21$$
$$72=8x$$
$$x=9$$

2. A number is cut in half and is then added to $\frac{8}{3}$. This sum is equal to the difference of $\frac{3}{5}$ of the number and $\frac{9}{4}$. What is the number?

SOLUTION:

$$\frac{x}{2}+\frac{8}{3}=\frac{3}{5}x-\frac{9}{4}$$
$$30x+160=36x-135$$
$$295=6x$$
$$x=\frac{295}{6} \text{ or } x=49\frac{1}{6}$$

3. Find the solution(s) to the following equation:

|x|+|x+6|=10

SOLUTIONS:

<u>Analytical Solution</u>: This equation defines a piecewise linear function:

1.On the interval (-∞,-6], we have y = -2x -6

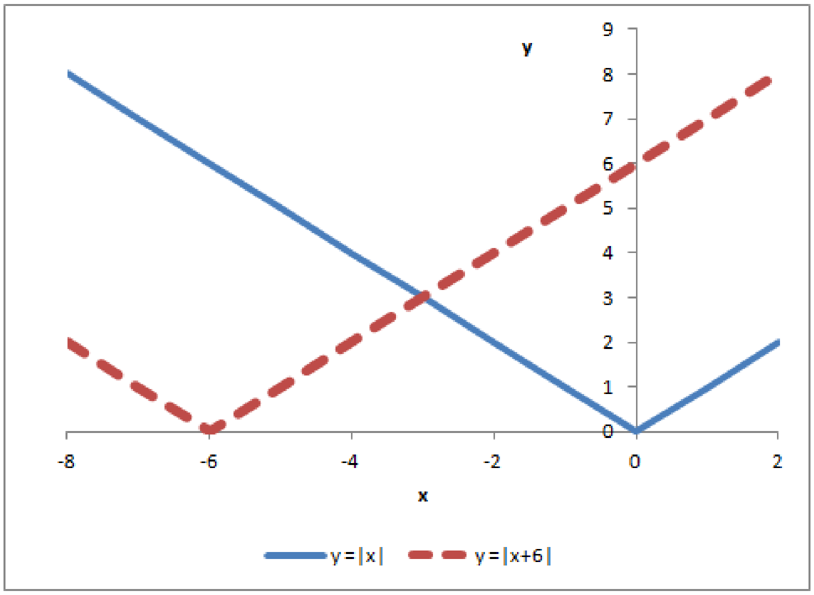
2.On the interval (-6,0], we have y = (x + 6) - x → y = 6

3.On the interval (0,∞), we have y = 2x + 6

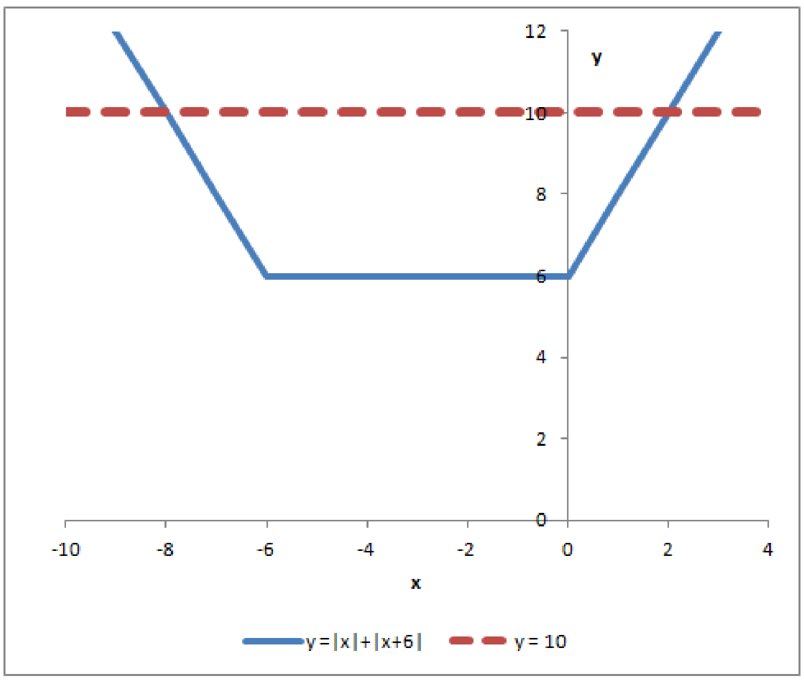
Therefore, we need only find the values of x in each of the three above equations where y is equal to 10. In the first equation, we see that x= -8 is a solution. There are no solutions in the second equation since y is constant. Finally, the third equation gives us that x=2 is the only remaining solution. Hence, **x = {-8, 2}. Both solutions required to receive credit.**

Graphical Solution: To find the solutions, we will plot the function y = |x|+|x + 6| and then determine where the graph intersects the horizontal line y = 10. However, it is not easy to draw this graph directly without the use of a calculator.

To avoid this, we can simply draw the graphs of y = |x| and y = |x + 6| and then add them to get the desired result. Here are the plots of y = |x| and y = |x + 6|



We add the two graphs together and then determine where their sum equals 10



It is readily seen that the solution set is given by **x={-8, 2}. Note that both solutions are necessary for credit on the problem.**

Round 3: Logic Problems

1. What is the sum of all the integers from -17 to +20?

Solution: The sum of the integers from -17 to 17 is equal to 0. Then 18 + 19 + 20 = **57**.

2. Several people are in a line. Starting from one end Jane is counted as the 4th person in line and starting from the other end she is counted as the 13th person in line. How many total people are in the line?

Solutions:
Counting Method: There are three people standing in front of Jane and twelve standing behind her. Therefore, there are 3 + 1 + 12 = **16 people** standing in the line.

Visual Method: An alternative visual solution is to draw a diagram where each space represents a person in line: _ _ _ (Jane) _ _ _ _ _ _ _ _ _ _ _ _ Counting all the spaces including Jane we see that there are **16 people** in the line.

3. Old MacDonald had a farm, and he also had a love of logic puzzles. Using a code where each letter represents a different single digit, he noticed that COW = 306 and that

```
 GOOSE
- DUCK
   EGG
```

What is the four-digit number represented by the word DUCK?

Solution: Since we are subtracting a 4-digit number from a 5-digit number and ending with a 3 digit result, we have that G=1 and D=9. With this information, we plug in G=1 into EGG to deduce that S is either 4 or 5, since S - 3 = 1 (S is 5 if it is "borrowed from" in the units column). But if S was borrowed from, then E would have to be 0 (and K would be 9) to make their difference 1 in the ones place. But since O is already 0, E can't be, so S can't be 5. Therefore, we conclude that S=4.

Since G=1 and O=0(zero), we know that K and E must be consecutive increasing integers. The only remaining consecutive increasing integers are 7 and 8, so K=7 and E=8.

Finally, since O=0(zero) and E=8, we have that U=2, giving us the answer that **DUCK = 9237**.

Round 4: Ratios, Proportions, and Variation

1. If 8 is 40% of x and 12 is 80% of y, then y is what percentage of x?

Solution:

$$0.4x = 8 \qquad 0.8y = 12$$
$$x = \frac{8}{0.4} \qquad y = \frac{12}{0.8}$$
$$x = \frac{2}{0.1} = 20 \qquad y = \frac{3}{0.2} = 15$$

Now that we know the values of x and y, we need to compute what percentage of x is y.

Method 1 (Direct Computation):

$$\frac{y}{x} = \frac{15}{20} = \frac{3}{4} = 75\%$$

Method 2 (Algebraic Word Problem)

An alternative method of finding the percentage would be to express the following algebraic problem: what percentage of 20 is 15? Letting z be the unknown percentage, this problem can be written as

$$15 = \left(\frac{z}{100}\right)20$$
$$1500 = 20z$$
$$z = 75$$

2. Suppose that c is directly proportional to the cube of d and inversely proportional to the square of e. If c = 32 when d = 4 and e = 6, find the value of c when d = 6 and e = 12.

Solution: If we let k be the unknown constant of proportionality, then the above description can be expressed

$$32 = k\frac{4^3}{6^2}$$
$$32 = k\frac{64}{36}$$
$$32 = k\frac{16}{9}$$
$$2 = k\frac{1}{9}$$
$$9 \times 2 = k$$
$$k = 18$$

Now that we know k, we can plug in the new values of d and e to determine c:

$$
\begin{aligned}
c &= 18 \cdot \frac{6^3}{12^2} \\
c &= 18 \cdot \frac{6^2 \times 6}{(6 \times 2)^2} \\
c &= 18 \cdot \frac{6^2 \times 6}{6^2 \times 2^2} \\
c &= \frac{18}{2} \times \frac{6}{2} = 9 \times 3 = 27
\end{aligned}
$$

3. There are 2820 tractors in Farmtown. The ratio of the number of medium tractors to the number of small tractors is 7:5 and the ratio of the number of small tractors to the number of large tractors is 8:9. How many large tractors are there in Farmtown?

Solutions:

Method 1: We need to find a way to express the ratio between the numbers of all three different size tractors. Note that the ratio 7:5 (# of medium tractors to # of small tractors) can be rewritten as 56:40. Further, the ratio 8:9 (# of small tractors to # of large tractors) can be rewritten 40:45.

Therefore, we know that for every set of 40 small tractors, there must be a corresponding set of 56 medium tractors and also a set of 45 large tractors. If there are 2820 tractors in total, we can use algebra to determine how many sets x of 40 small tractors there are:

$$
\begin{aligned}
&40x + 56x + 45x = 2820 \\
&141x = 2820 \\
&x = 20
\end{aligned}
$$

Now that we know there are 20 sets of 40 small tractors in Farmtown, we know there must be 20 sets of 45 large tractors as well. This means there are **900 large tractors.**

Method 2 (More Visual):
An alternative way of visualizing the ratios in this problem is as follows:

Small	:	Medium	:	Large	
5	:	7			<--- Multiply this line by 8
8			:	9	<--- Multiply this line by 5
40	:	56	:	45	

The remainder of the problem is solved the same way as in method 1.

Team Round

1. If distinct beads A,B,C,D and E are strung on a bracelet, ABCDE, BCDEA and EDCBA are all orderings for the same stringing. How many different stringings can be made using these five beads?

Solutions:

<u>Listing</u>: As the statement of the problem hints, we need to account for rotational symmetry <u>and</u> flipping symmetry in this problem in order to avoid double-counting stringings. The simplest way to do this is to fix one particular bead as the first bead in every stringing (in the solution below we have chosen bead A for this role). Then we need only enumerate all possible ways to arrange the four beads after bead A (while remembering to remove mirror images), which gives **<u>12 stringings in total:</u>**

ABCDE	ACBDE	ADBCE
ABCED	ACBED	ADCBE
ABDCE	ACDBE	
ABDEC	ACEBD	
ABECD		
ABEDC		

<u>Counting Argument</u>: The total number of ways we can <u>arrange</u> 5 beads is given by 5! = 120. Now note that each distinct stringing can be represented by 10 different <u>arrangements</u> (each bracelet can be rotated to give 5 different arrangements and then flipped to give 5 more). Hence, the total number of distinct stringings is 120/10 = **<u>12</u>.**

2. The perimeter of a rectangle is 20 units, and the length of the rectangle is 1.75 times the width. If we increase the perimeter of the rectangle to 30 units while keeping the length-to-width proportion constant, by what percentage does the area of the rectangle increase?

Solutions:

<u>Computation (decimals)</u>: Let l denote length and w denote width. We have that $2l + 2w = 20$ and that $l = 1.75w$. This gives that $5.5w = 20 \Rightarrow w = 3.636$. Therefore $l = 6.364$ and the original area = 23.140 sq units. If perimeter increases to 30 with no change in proportion, the equations now give that $5.5w = 30 \Rightarrow w = 5.455$ and so $l = 9.546$. This means the new area is 52.065 sq units. 52.065 divided by 23.140 is equal to 2.25, meaning that a **<u>125% increase</u>** in area has occurred.

<u>Computation (fractions)</u>: Let l denote length and w denote width. We have that $2l + 2w = 20$ and that $l = \frac{7}{4}w$. This gives that $\frac{11}{2}w = 20 \Rightarrow w = \frac{40}{11}$. Therefore $l = \frac{7}{4} \times \frac{40}{11}$ and the original area = $\left(\frac{7}{4} \times \frac{40}{11}\right) \times \left(\frac{40}{11}\right)$ sq units. If perimeter increases to 30 with no change in proportion, we can conclude immediately that the new width and length are $\frac{3}{2}$ times the originals, giving that the new area is

$\left(\frac{3}{2}\times\frac{7}{4}\times\frac{40}{11}\right)\times\left(\frac{3}{2}\times\frac{40}{11}\right)$ sq units. Dividing the new area by the old, we see that the ratio is equal to $\frac{3}{2}\times\frac{3}{2}=\frac{9}{4}=2.25$, which represents a **125% increase** in area.

Proportions Method: A fast solution to the problem is to note that the area of the rectangle is given by wl. Since the proportions of the rectangle do not change, the increase in perimeter by 50% increases both width and length by precisely 50% as well. Hence the ratio of the new area to the old area is given by

$$\frac{1.5w\times1.5l}{wl}=1.5^2=2.25$$

And so area has **increased by 125%**.

3. If $a\diamond b=a^b$ and $a\star b=ab^2$, evaluate

$$128\diamond\frac{(16\star2)\diamond\frac{1}{3}}{7\star\left(2048\diamond\frac{1}{11}\right)}$$

Solution:

$$=128\diamond\frac{\sqrt[3]{(16\times2^2)}}{7\times\left(\sqrt[11]{2048}\right)^2}$$

$$=128\diamond\frac{\sqrt[3]{64}}{7\times2^2}$$

$$=128\diamond\frac{4}{28}$$

$$=\sqrt[7]{128}$$

$$=2$$

4. Suppose that the letters A,B,C,D and E each represent a distinct digit. Determine which digit each letter must be if

$$\begin{array}{r} \text{A B C D E} \\ \times \qquad\quad 4 \\ \hline \text{E D C B A} \end{array}$$

Solution: A can be at most 1 or 2 since 4 times A plus potentially a carry must be less than 9. If A is 1 there is no number E such that 4 times E will be equal 1, so we conclude that A is 2.

Since A is 2, E is either 8 or 9, depending on if there is a carry. But when we evaluate the ones unit, we see that E is either 3 or 8 given that A is 2. Therefore, E must be 8.

If B were any larger than 2, it would ensure that we would have to carry over a digit from the fourth column to the fifth. But we know there is no carry there since A = 2 and E = 8. Therefore, B can either be 1 or 2. We already concluded that A is 2, so B is 1.

Since E is 8, we know the carry in the second column is three. Further, since B is 1, the units digit of the number 3 + 4D must be 1. The only possible candidates for D then are 2 or 7, but we already established A is 2, so D is 7.

Since D is 7 and the carry in the second column is 3, the carry in the third column must be 3 as well. Therefore, the units digit of the number 3 + 4C must be C. Of all the remaining digits left unassigned, the only one which satisfies this equation is C = 9.

Therefore, the answer is **ABCDE = 21978**.

5. 17 apples cost as much as 6 cherries. 18 cherries cost as much as 34 bunches of grapes. Kate spent twice as much money on grapes as she did on apples. If she bought 8 bunches of grapes and no cherries, how many apples did Kate buy?

Solution: From the price ratios we determine that 51 apples cost as much as 34 bunches of grapes. This simplifies to 3 apples costing the same as 2 bunches of grapes. If Kate bought 8 bunches of grapes and she spent twice as much money on them as she did on apples, then she must have purchased 6 apples.

6. Solve for y in terms of x and simplify:

$$2y+2x=3(4-6yx)-4(-x)-12y$$

Solution: Begin by expanding all terms. Then move all terms containing y to the left side and all other terms to the right side:

$$2y+2x=12-18xy+4x-12y$$
$$2y+18yx+12y=12+4x-2x$$
$$y(14+18x)=12+2x$$
$$y=\frac{12+2x}{14+18x}$$
$$y=\frac{6+x}{7+9x}$$

7. Michael is walking from his home to his school. At 8:10, he has walked one fourth of the total distance. At 8:12, he has walked one third of the total distance. Assuming Michael walks at a constant rate of speed, at what time did Michael leave his home?

Solution:
Michael walks one twelfth of the total distance (one third minus one fourth) in 2 minutes, which implies that it is a 24 minute walk. At 8:10 he has walked one fourth of the distance (and also one fourth of the total time) meaning he has been walking for 6 minutes. Therefore he must have left home at 8:04.

8. Jake took an exam that had 200 multiple choice questions. Each correct answer earned him 4 points and each incorrect answer deducted 2 points from his score. If Jake answered every question on the exam and he received a total score of 536, how many questions did he answer incorrectly?

Solutions:
<u>Trial and Error</u>:

Correct		Wrong	Score	
200*4	+	0 * (-2)	= 800	
100*4	+	100* (-2)	= 400 - 200 = 200	
150*4	+	50 *(-2)	= 600 - 100 = 500	
160*4	+	40* (-2)	= 640 - 80 = 560	
155*4	+	45 *(-2)	= 620 - 90 = 530	
156*4	+	44*(-2)	= 624 - 88 = 536	→ **He answered 44 questions incorrectly**

<u>Computational</u>: A total of 800 points is possible. By answering a question correctly, Jake loses no points. By answering a question incorrectly, he effectively loses 6 points since he misses the chance to receive 4 points and then also suffers a 2 point deduction. Therefore, we can solve for the number of incorrect answers by dividing the total number of missed points by 6:

$$\frac{800-536}{6} = 44$$