Freshman Meet 3 – February 12, 2014
Round 1: Graphing on a Number Line

NO CALCULATOR ALLOWED

Draw the graph of each of the following inequalities on the corresponding number line provided below. Please specify all endpoints on your graph. Your graph need not be drawn to scale.

1. $4x + 8 \geq 5x - 5$

2. $|2x + 3| \geq 11$

3. $\dfrac{4}{x} < x$

ANSWERS

(1 pt.) 1.

(2 pts.) 2.

(3 pts.) 3.

Freshman Meet 3 – February 12, 2014
Round 2: Operations on Polynomials

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Let x, y, and z be variables. What is the degree of $-6xy^3z^4$?

2. Determine the value of m such that $2x^2 - 7x + m$ is divisible by $x - 3$.

3. Solve for all possible values of x:

$$(3x-5)(2x+7)-7=(2x-1)^2+(x-3)(2x+1)+5$$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. $m =$ ______________________

(3 pts.) 3. $x =$ ______________________

Freshman Meet 3 – February 12, 2014
Round 3: Techniques of Counting and Probability

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Suppose you open a pizza parlor and decide to offer your pizzas in a) small, medium, or large; b) deep dish or thin crust; and c) with or without extra cheese. Given only these options, how many different kinds of pizzas are possible?

2. A woman and her husband board an airplane with their 4 children: 2 boys and 2 girls. They are assigned a row of seats consisting of three seats on either side of the aisle. If the parents occupy the two aisle seats, what is the probability, if the children are randomly seated, that the two girls will be next to each other?

3. If a number is chosen at random from all six-digit numbers that contain each of the digits 1, 2, 3, 4, 5, and 6 exactly once (ex. 653214), what is the probability that it contains the three-digit block 123 (ex. 6<u>123</u>54)?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Freshman Meet 3 – February 12, 2014
Round 4: Perimeter, Area, and Volume

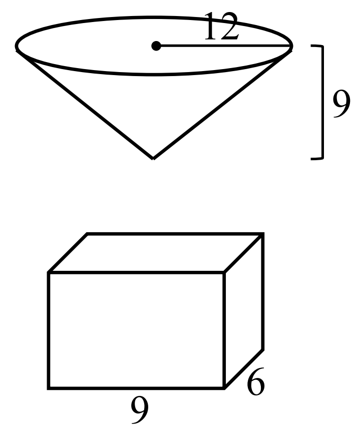
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. The area of a triangle is 96 square centimeters. If the base has length 12 cm, find the altitude of the triangle in centimeters.

2. The circumference of circle B is twice the circumference of circle A. The sum of the areas of both circles is 25π. What is the radius of circle B?

3. In the diagram shown, the right circular cone of radius 12 inches and height 9 inches is initially filled with water. The water from the cone is drained into the initially empty rectangular prism below it until the surface of the water is halfway down the lateral edge of the cone. How high, ***to the nearest inch***, will the water be in the rectangular prism? ($\pi \approx 3.1416$)



ANSWERS

(1 pt.) 1. ________________ cm

(2 pts.) 2. ________________ units

(3 pts.) 3. ________________ inches

Freshman Meet 3 – February 12, 2014
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. Determine the x-intercept of $\overleftrightarrow{AB}$ if $A = (-3, 8)$ and $B = (4, -20)$.
2. Three times the width of a rectangle exceeds twice its length by 3 inches. Four times its length is 12 more than its perimeter. Find the area of the rectangle, in square inches.
3. In 1770, Joseph-Louis Lagrange proved that every positive integer can be written as the sum of at most four perfect squares. Find the smallest integer ≥ 80 that cannot be written as the sum of 3 perfect squares.
4. The sum of two numbers is 37. If the larger is divided by the smaller, the quotient is 3 and the remainder is 5. Find the larger number.
5. Given that
$$x^4 + 2x^2 + 2 = y^4 - 4y^3 + 6y^2 - 4y + 2,$$
find all possible real values of y if $x = \sqrt{3}$.
6. If $\dfrac{5x+5}{x^2 - x - 6} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$, find $A + B$.
7. In how many distinguishable ways can four identical red chips and two identical white chips be arranged, equally spaced, around a circle? Arrangements related by rotation are not distinguishable.
8. Draw the graph of the following inequality of the number line provided on the answer sheet. Please specify all endpoints on your graph. Your graph need not be drawn to scale.
$$2x + |3x + 5| \leq 9$$

Freshman Meet 3 – February 12, 2014
TEAM ROUND ANSWER SHEET

1. ______________________

2. ______________________ in^2

3. ______________________

4. ______________________

5. $y =$ ______________________

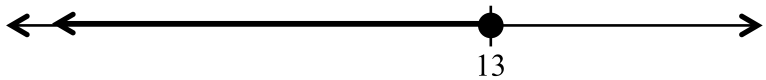
6. ______________________

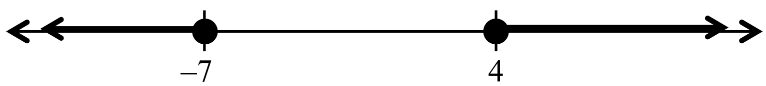
7. ______________________

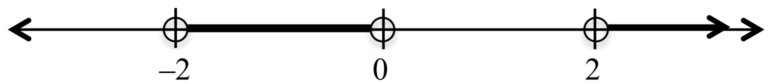
8. ⟷

Freshman Meet 3 – February 12, 2014
ANSWERS

ROUND 1
(Millbury, Southbridge, Worc Acad)

1. 

2. 

3. 

ROUND 2
(Shrewsbury, Notre Dame, Doherty)

1. 8
2. $m = 3$
3. $x = 9/4 = 2\frac{1}{4} = 2.25$

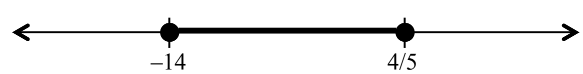
ROUND 3
(Shrewsbury, Shrewsbury, Westborough)

1. 12
2. $1/3 = 0.\overline{3} = 33.\overline{3}\% = 33\frac{1}{3}\%$
3. $1/30 = 0.0\overline{3} = 3.\overline{3}\% = 3\frac{1}{3}\%$

ROUND 4
(W Boylston, Quaboag, Bancroft)

1. 16 cm
2. $2\sqrt{5}$
3. 22 in

TEAM ROUND
(Hudson, Nashoba, QSC, Leicester, AMSA Charter, Auburn, Hudson?, Algonquin)

1. -1
2. 315 in^2
3. 87
4. 29
5. $y = 3, -1$ (need both, either order)
6. 5
7. 3
8. 

Freshman Meet 3 – February 12, 2014
FULL SOLUTIONS

ROUND 1

1. We have

$$\begin{aligned} 4x+8 &\geq 5x-5 \\ 13 &\geq x. \end{aligned}$$

2. The absolute value sign gives rise to two cases.

   First case: $2x+3 \geq 11 \implies x \geq 4$.

   Second case: $2x+3 \leq -11 \implies x \leq -7$.

3. We find the critical points of the inequality. These are at points of equality ($x = \pm 2$) and points where any expression in the inequality is undefined ($x = 0$).

   We now must test points in each region. Convenient choices are -4, -1, 1, and 4.

   $x = -4$:

$$\begin{aligned} \frac{4}{-4} &\ ? \ -4 \\ -1 &> -4 \end{aligned}$$

   $x = -1$:

$$\frac{4}{-1} < -1$$

   $x = 1$:

$$\frac{4}{1} > 1$$

   $x = 4$:

$$\begin{aligned} \frac{4}{4} &\ ? \ 4 \\ 1 &< 4 \end{aligned}$$

   Therefore, our answer is $-2 < x < 0$ or $x > 2$.

ROUND 2

1. The degree is 1 in x, 3 in y, and 4 in z, so the degree of the product is $1+3+4=\boxed{8}$.

2. **METHOD I:** Using the REMAINDER THEOREM, we know that $2(3)^2-7(3)+m=0$, giving $m=\boxed{3}$.

   METHOD II: Use long division to divide $2x^2-7x+m$ by $x-3$:

$$\begin{array}{r|l} & 2x-\ \ 1 \\ x-3 & 2x^2-7x+m \\ & \underline{2x^2-6x} \\ & \quad -x+m \\ & \quad \underline{-x+3} \\ & \qquad m-3 \end{array}$$

   Therefore, $m-3=0$, giving $m=\boxed{3}$.

3. Expand the polynomial products.

$$\begin{aligned} (3x-5)(2x+7)-7 &= (2x-1)^2+(x-3)(2x+1)+5 \\ (6x^2+11x-35)-7 &= (4x^2-4x+1)+(2x^2-5x-3)+5 \\ 6x^2+11x-42 &= 6x^2-9x+3 \\ 20x &= 45 \\ x &= \boxed{9/4}. \end{aligned}$$

ROUND 3

1. There are 3 options for size, 2 options for crust, and 2 options for cheese, giving a total of $3\cdot 2\cdot 2=\boxed{12}$ options for the pizza.

2. **METHOD I:** Consider the position of one of the girls. The seat next to her is equally likely to be occupied by her sister, brother A, or brother B. Therefore, there is a $\boxed{1/3}$ probability that it is her sister (and therefore the two girls are seated next to each other).

   METHOD II: There are a total of $4!=24$ ways for the four children to occupy the four seats. If the two girls are seated next to each other, they can be on either side of the aisle (2 ways), the girls can switch seats (2 ways), and the boys can switch seats (2 ways). Therefore, a total of $2\cdot 2\cdot 2=8$ configurations have the girls seated next to each other. The probability is hence $8/24=\boxed{1/3}$.

3. A total of $6! = 720$ such six-digit numbers exist. The 123 block can be in any of 4 possible positions. In each position, the other three digits can be arranged in $3! = 6$ different ways. Therefore, the required probability is $\frac{4 \cdot 6}{720} = \boxed{\frac{1}{30}}$.

ROUND 4

1. The area of a triangle is one-half the product of the base and altitude. Therefore, the altitude is $96 \div 12 \times 2 = \boxed{16}$.
2. If the circumference of circle B is twice the circumference of circle A, then the area of circle B is $2^2 = 4$ times the area of circle A. Therefore, circle B has area 20π. Since the area of a circle of radius r is πr^2, the radius of circle B must be $\sqrt{20} = \boxed{2\sqrt{5}}$.
3. The volume of the cone is $\frac{1}{3}\pi r^2 h = 432\pi$. The volume of the rectangular prism filled to a height H is $6 \cdot 9 \cdot H = 54H$. If the water in the cone is drained such that the surface of the water is halfway down the lateral edge, the proportion of the volume drained is $1 - \left[\frac{1}{2}\right]^3 = \frac{7}{8}$. Equating volumes to solve for H,

$$\begin{aligned} \frac{7}{8} \cdot 432\pi &= 54H \\ 378\pi &= 54H \\ \frac{378}{54}\pi &= H \\ 7\pi &= H. \end{aligned}$$

To the nearest integer, $7\pi \approx \boxed{22}$.

(The arithmetic is easier if we notice that the height of the cone is the same as one of the dimensions of the base of the rectangular prism. Then, the volume of the cone is $\frac{1}{3}\pi(12^2)(9) = 48 \cdot 9\pi$ and the volume of the rectangular prism filled to a height H is still $6 \cdot 9 \cdot H$. Cancel out the factor of 9 to get that $48\pi = 6H$ or $H = 8\pi$ if the cone is drained entirely. Since it is only drained 7/8 of the way, $H = 7\pi \approx \boxed{22}$, as before.

TEAM ROUND

1. The slope of the line is $\frac{28}{-7} = -4$, and the slope-intercept form is $y = -4x - 4$. The x-intercept is the value of x when $y = 0$. Setting $-4x - 4 = 0$, we find that $x = \boxed{-1}$.

2. We have the system

$$\begin{cases} 3w &= 2l+3 \\ 4l &= 2l+2w+12 \end{cases}$$

Therefore, $2w+12 = 2l = 3w-3$, so $w = 15$. Plugging back in, $l = 21$. The area of the rectangle is $lw = 21 \cdot 15 = \boxed{315}$.

3. We have, for example:

$$\begin{aligned} 80 &= 8^2+4^2 \\ 81 &= 9^2 \\ 82 &= 9^2+1^2 \\ 83 &= 9^2+1^2+1^2 \\ 84 &= 8^2+4^2+2^2 \\ 85 &= 9^2+2^2 \\ 86 &= 9^2+2^2+1^2 \\ 87 &= 9^2+2^2+1^2+1^2 \\ &= 7^2+6^2+1^2+1^2 \end{aligned}$$

so $\boxed{87}$ is the smallest integer ≥ 80 that cannot be expressed as the sum of 3 squares. Many integers can be represented as the sum of 4 squares <u>in more than one way</u>, possibly using zero as one or more of the squares.

[NOTE: You do not need to look far to find an integer that cannot be expressed as the sum of ≤ 3 perfect squares. Consider the perfect squares modulo 8: the only values are 0, 1, and 4. Therefore, any number congruent to 7 mod 8 cannot be written as the sum of ≤ 3 squares. In fact, all integers of the form $4^n(8k+7)$ for $n, k \in \mathbb{Z}$ cannot be written as the sum of ≤ 3 squares.]

4. Let the smaller number be x. Since the larger number gives a quotient of 3 and a remainder of 5 when divided by the smaller number, the larger number must be $3x+5$. Therefore, $x+(3x+5) = 37$ and so $x = 8$. The larger number is $3 \cdot 8 + 5 = \boxed{29}$.

5. Subtracting 1 from both sides and factoring,

$$(x^2+1)^2 = (y-1)^4.$$

Since $x = \sqrt{3}$, the LHS is equal to 16. Since y is real, $(y-1)^2 = 4$ (reject the negative square root -4). Therefore, $y-1 = \pm 2$ so $y = \boxed{3, -1}$.

6. To solve the equation
$$\frac{5x+5}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}$$
for A and B, cross-multiply:
$$\frac{5x+5}{x^2-x-6}=\frac{A(x+2)+B(x-3)}{x^2-x-6}.$$
Equate the coefficients for x and the constant term. This leaves the system of equations
$$\begin{cases} A+B &= 5 \\ 2A-3B &= 5. \end{cases}$$
Solving gives $(A,B)=(4,1)$, so the sum $A+B$ is $\boxed{5}$.

[This process is called Partial Fraction Decomposition.]

7. The white chips can either be adjacent, separated by 1 red chip, or separated by 2 red chips. No other configurations are possible. Therefore, there are $\boxed{3}$ distinguishable arrangements.

8. Isolate the absolute value and solve:
$$\begin{array}{rcccl} & & 2x+|3x+5| & \leq & 9 \\ & & |3x+5| & \leq & 9-2x \\ 2x-9 & \leq & 3x+5 & \leq & 9-2x \\ 2x-14 & \leq & 3x & \leq & 4-2x \end{array}$$
The left inequality gives $x \geq -14$ and the right inequality gives $5x \leq 4$ or $x \leq 4/5$. These inequalities are related by an AND relationship, not an OR relationship. Therefore, the answer is $-14 \leq x \leq 4/5$.