Worcester County Mathematics League

Varsity Meet 4
March 1, 2017

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017

Round 1: Number Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. The GCF of 60 and another integer is 30 and their LCM is 180. Find the second integer.

2. How many different integers exist between 10^6 and 10^7 such that the sum of the integer's digits is equal to 2?

3. In hexadecimal, or base 16, the numbers 10 through 15 are represented by the letters A through F. Compute the following sum and express the answer as a base 10 number: $323_{16} + BA_{16} + 2D2_{16}$

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________

(3 pts.) 3. ______________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017

Round 2: Algebra I

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If the money in the teacher's pocket is divided evenly among the students in the room, each student would get \$1.47. If two more students arrived, each student would instead get \$1.05. How much money does the teacher have in her pocket?

2. One pump can fill a water tank in 3 hours and another takes 4 hours. When the tank is empty, both pumps are turned on for 40 minutes and then the faster pump is shut off. After this point, how much time does it take the slower pump to finish filling the tank?

3. Solve: $\frac{x+1}{x+2} + \frac{x+7}{x+8} = \frac{x+3}{x+4} + \frac{x+5}{x+6}$

ANSWERS

(1 pt.) 1. \$________________________

(2 pts.) 2. ________________________ hours

(3 pts.) 3. ________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

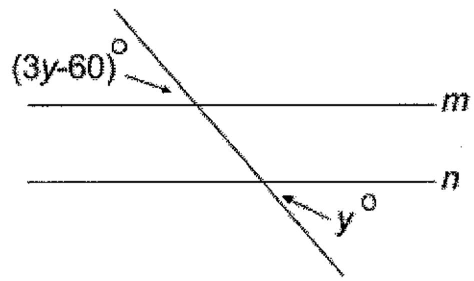
Varsity Meet 4 - March 1, 2017

Round 3: Geometry

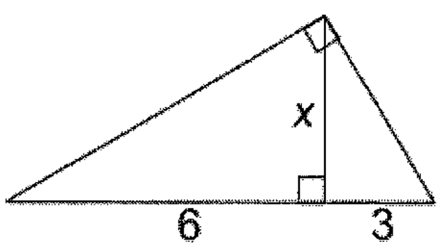
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $m \parallel n$, find y. Note that the diagram is not drawn to scale.



2. Find the value of x :



3. A circle passes through all 3 vertices of a triangle with sides 7.5, 10 and 12.5 inches. What is the circumference of the circle?

ANSWERS

(1 pt.) 1. ________________ degrees

(2 pts.) 2. ________________ units

(3 pts.) 3. ________________ inches

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017

Round 4: Logs, Exponents, and Radicals

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve for x: $log_3 x + log_3(x-6) = 3$

2. Solve for x: $\left(\frac{1}{3}\right)^{-1} \cdot (9)^{x-1} = \left(\frac{1}{9}\right)^{2x-1}$

3. If $y = log_4 x$, find the real value of x which satisfies $log_y\left(log_4 x^2\right) = 2$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017

Round 5: Trigonometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $\tan x + \cot x = \frac{144}{25}$, what is the value of $\frac{1}{\tan x} + \frac{1}{\cot x}$?

2. Evaluate: $(\tan 10^\circ)(\tan 20^\circ)(\tan 30^\circ)\cdots(\tan 80^\circ)$

3. Express the following quantity as a single trigonometric function:
$(\sin 2x)(\csc x) - (\cos 2x)(\sec x)$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017

Team Round

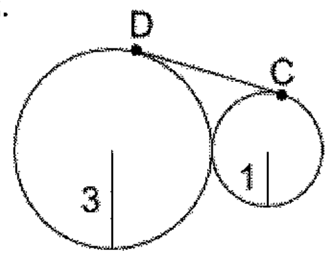
All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 points each)

APPROVED CALCULATORS ALLOWED

1. Determine the digit A such that the six digit number $3730A5$ is divisible by 21.

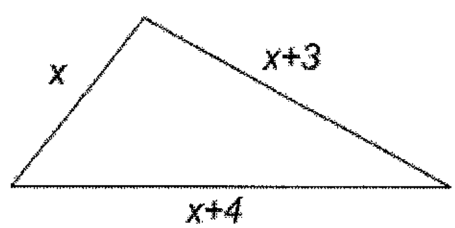
2. The period of a wave is the reciprocal of the wave's frequency, f. A test device emits sound waves which are linearly increasing in frequency over time. At time $t = 0$ sec, $f = 240$ Hz. At $t = 2$ min 40 sec, $f = 400$ Hz. What is the period of the sound waves emitted by the device at $t = 1$ min 25 sec?

3. If the two circles in the diagram are tangent to one another, find the length of the external tangent segment DC.



4. Solve for x: $216^{\frac{x-3}{3}} = 36^{\frac{2}{x}}$.

5. Find $x > 0$ if the cosine of the smallest angle of this triangle is $\frac{5}{6}$.



6. If $log_2\left(log_3\left(log_x(3)\right)\right) = 1$ what is $log_3 x$?

7. A fair coin is flipped 5 times. What is the probability that at least 2 heads are flipped?

8. What is the sum of all the coefficients in the expansion of $(23x - 16y)^{10}$?

9. Suppose there is a function f such that $\frac{f(x^2+1)}{x} = k$ for some real number k and for all $x \neq 0$. If $m \neq 0$, what is $\frac{f\left(\frac{m^2+16}{16}\right)}{m}$ in terms of k?

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017
Team Round Answer Sheet

1. ______________________

2. ______________________ seconds or Hz^{-1}

3. ______________________ units

4. ______________________

5. ______________________ units

6. ______________________

7. ______________________

8. ______________________

9. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017 - ANSWER KEY

Round 1:

1. 90 (Groton-Dunstable)
2. 7 (Shepherd Hill)
3. 1711 (Shepherd Hill)

Round 2:

1. 7.35 (Mass Academy)
2. $\frac{22}{9}$ or $2\frac{4}{9}$ or $2.\overline{4}$ (Groton-Dunstable)
3. -5 (Worcester Academy)

Round 3:

1. 30 (Hudson Catholic)
2. $3\sqrt{2}$ (Douglas)
3. $\frac{25}{2}\pi$ or $12\frac{1}{2}\pi$ or 12.5π (Auburn)

Round 4:

1. 9 (West Boylston)
2. $\frac{1}{2}$ or 0.5 (Burcoat)
3. 16 (Westborough)

Round 5:

1. $\frac{144}{25}$ or $5\frac{19}{25}$ or 5.76 (Houghton)
2. 1 (St. John's)
3. $\sec x$ or $\frac{1}{\cos x}$ (Northbridge)

TEAM Round

1. 6 (Algonquin)
2. $\frac{1}{325}$ or 0.00301 (Leicester)
3. $2\sqrt{3}$ (St. John's)
4. -1 or 4 (Both required) (Shrewsbury)
5. 5 (Worcester Academy)
6. $\frac{1}{9}$ or $0.\overline{1}$ (Leicester)
7. $\frac{13}{16}$ or 0.813 (Notre Dame Academy)
8. 7^{10} (St. John's)
9. $\frac{1}{4}k$ or $0.25k$ (Burncoat)

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 - March 1, 2017 - SOLUTIONS

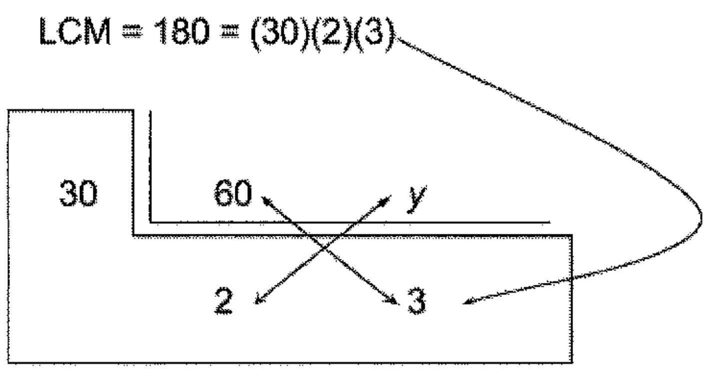
Round 1: Number Theory

1. The GCF of 60 and another integer is 30 and their LCM is 180. Find the second integer.

Solution 1: We begin by decomposing 60 into its prime factorization: $60 = 2^2 \times 3 \times 5$. We have that the greatest common factor of the two integers is 30, which is decomposed as $2 \times 3 \times 5$. Finally, we have that the least common multiple of the two integers is $180 = 2^2 \times 3^2 \times 5$.

Therefore, the second number must be $\frac{(2^2 \times 3^2 \times 5) \times (2 \times 3 \times 5)}{2^2 \times 3 \times 5} = 2 \times 3^2 \times 5 = 90$.

Solution 2: Let y be the second integer. We can determine its value using the following diagram:



$(30)(2)(3) = 180$
$(3)(60) = 180$
$(2)(y) = 180 \longrightarrow y = 90$

2. How many different integers exist between 10^6 and 10^7 such that the sum of the integer's digits is equal to 2?

Solution 1 (Extrapolation): Begin by examining smaller powers of 10. Notice that between 10^1 and 10^2, there are exactly two numbers satisfying the condition: 11 and 20. Similarly, between 100 and 1000 (or 10^2 and 10^3) there are 3 numbers satisfying the condition: 101, 110 and 200. Therefore, between 10^6 and 10^7 there are 7 numbers satisfying the condition.

Solution 2 (Brute Force): $10^6 = 1,000,000$ and $10^7 = 10,000,000$. Between these two numbers, we have that the numbers 1,000,001 1,000,010 1,000,100 1,001,000 1,010,000 1,100,00 and 200,000 all satisfy the condition, for seven numbers in total.

3. In hexadecimal, or base 16, the numbers 10 through 15 are represented by the letters A through F. Compute the following sum and express the answer as a base 10 number: $323_{16} + BA_{16} + 2D2_{16}$

Solution 1: Compute the total sum by adding two terms at a time. In base 16, we have that

$$\begin{array}{cccc} & 3 & 2 & 3 \\ + & & B & A \\ \hline & 3 & D & D \end{array}$$

Next we have that

$$\begin{array}{cccc} & 1 & & \\ & 3 & D & D \\ + & 2 & D & 2 \\ \hline & 6 & A & F \end{array}$$

We finally convert this number to base 10 by

$$6AF_{16} = 6(16^2) + 10(16) + 15(1)$$
$$6AF_{16} = 6(256) + 160 + 15$$
$$6AF_{16} = 1536 + 175 = 1711.$$

Solution 2: Converting from hexadecimal to decimal, we have that

A = 10
B = 11
C = 12
D = 13
E = 14
F = 15

Now convert each term of the sum into decimal:

$$323_{16} + BA_{16} + 2D2_{16}$$
$$\left(3(16^2) + 2(16) + 3(1)\right) + (11(16) + 10(1)) + \left(2(16^2) + 13(16) + 2(1)\right)$$

Combining like terms we have

$$(3+2)(16^2) + (2+11+13)(16) + (3+10+2)(1)$$
$$5(256) + 26(16) + 15$$
$$1280 + 416 + 15 = 1711.$$

Round 2: Algebra I

1. If the money in the teacher's pocket is divided evenly among the students in the room, each student would get \$1.47. If two more students arrived, each student would instead get \$1.05. How much money does the teacher have in her pocket?

Solution: Let x be the original number of students in the class and y be the amount of money in the teacher's pocket. We have that

$$\frac{y}{x} = 1.47 \quad \Rightarrow \quad y = 1.47x \qquad (1)$$
$$\frac{y}{x+2} = 1.05 \quad \Rightarrow \quad y = 1.05(x+2) \qquad (2)$$

Notice that the left hand sides of equations (1) and (2) are equal, which means that their right hand sides must also be equal. Therefore,

$$1.47x = 1.05(x+2)$$
$$1.47x = 1.05x + 2.10$$
$$0.42x = 2.10$$
$$x = 5$$

Since there are 5 students in the original class, we have the amount of money in the teacher's pocket is equal to $1.47 \times 5 = 7.35$

2. One pump can fill a water tank in 3 hours and another takes 4 hours. When the tank is empty, both pumps are turned on for 40 minutes and then the faster pump is shut off. After this point, how much time does it take the slower pump to finish filling the tank?

Solution 1: Let x be the desired amount of time. We have the following situation:

	Rate	$\times$	Time	=	Job Done
Pump 1 (fast)	$\frac{1}{3}$		$\frac{2}{3}$		$\frac{2}{9}$
Pump 2 (slow)	$\frac{1}{4}$		$\frac{2}{3}+x$		$\frac{2}{12}+\frac{x}{4}$

Since we know the slow pump runs until the tank is full, we have that

$$\frac{2}{9}+\frac{2}{12}+\frac{x}{4}=1$$
$$\frac{4}{18}+\frac{3}{18}+\frac{x}{4}=1$$
$$\frac{x}{4}=1-\frac{7}{18}$$
$$\frac{x}{4}=\frac{11}{18}$$
$$x=\frac{44}{18}=\frac{22}{9}.$$

Solution 2: Let x be the desired amount of time. After the first 40 minutes, we have the following situation:

	Rate	$\times$	Time	=	Job Done
Pump 1 (fast)	$\frac{1}{3}$		$\frac{2}{3}$		$\frac{2}{9}$
Pump 2 (slow)	$\frac{1}{4}$		$\frac{2}{3}$		$\frac{2}{12}$

This means that $\frac{2}{9}+\frac{2}{12}=\frac{4}{18}+\frac{3}{18}=\frac{7}{18}$ of the tank is full after the first 40 minutes. This leaves $\frac{11}{18}$ of the tank to go, so we simply need to determine how long it will take the slow pump to fill this amount.

Since the slow pump fills a quarter of a tank per hour, we have that it will take two hours for the pump to fill $\frac{1}{2}=\frac{9}{18}$ of the tank. To fill the final $\frac{2}{18}$ of the tank, it takes the pump $\frac{2}{18}\div\frac{1}{4}=\frac{4}{9}$ hours. Hence, it takes the slow pump a total of $2\frac{4}{9}$ hours to finish the job.

3. Solve: $\frac{x+1}{x+2} + \frac{x+7}{x+8} = \frac{x+3}{x+4} + \frac{x+5}{x+6}$

Solution 1: Notice that $\frac{x+1}{x+2} = 1 - \frac{1}{x+2}$. A similar expression holds for each of the other terms in the equation. Performing this substitution on each term gives

$$\left(1 - \frac{1}{x+2}\right) + \left(1 - \frac{1}{x+8}\right) = \left(1 - \frac{1}{x+4}\right) + \left(1 - \frac{1}{x+6}\right)$$

$$-\frac{1}{x+2} - \frac{1}{x+8} = -\frac{1}{x+4} - \frac{1}{x+6}$$

$$\frac{1}{x+2} + \frac{1}{x+8} = \frac{1}{x+4} + \frac{1}{x+6}$$

$$\frac{(x+2) + (x+8)}{(x+2)(x+8)} = \frac{(x+4) + (x+6)}{(x+4)(x+6)}$$

$$\frac{2x + 10}{(x+2)(x+8)} = \frac{2x + 10}{(x+4)(x+6)}$$

This equation holds if both sides are equal to 0, which would imply that $x = -5$, or when the denominators are equal. That is,

$$(x+2)(x+8) = (x+4)(x+6)$$
$$x^2 + 10x + 16 = x^2 + 10x + 24$$
$$16 = 24$$

Since 16 does not equal 24, we know that there is no x value such that the denominators will be equal. Therefore, we conclude that $x = -5$.

Solution 2 (Brute Force): Multiply both sides by the common denominator:

$$(x+2)(x+4)(x+6)(x+8)\times\left(\frac{x+1}{x+2}+\frac{x+7}{x+8}\right)=(x+2)(x+4)(x+6)(x+8)\times\left(\frac{x+3}{x+4}+\frac{x+5}{x+6}\right)$$

$$\frac{(x+1)(x+4)(x+6)(x+8)\ +\ (x+2)(x+4)(x+6)(x+7)}{(x+2)(x+4)(x+6)(x+8)}=\frac{(x+2)(x+3)(x+6)(x+8)\ +\ (x+2)(x+4)(x+5)(x+8)}{(x+2)(x+4)(x+6)(x+8)}$$

$$\frac{(x^4+19x^3+122x^2+296x+192)+(x^4+19x^3+128x^2+356x+336)}{(x+2)(x+4)(x+6)(x+8)}=$$

$$=\frac{(x^4+19x^3+124x^2+324x+288)+(x^4+19x^3+126x^2+344x+320)}{(x+2)(x+4)(x+6)(x+8)}$$

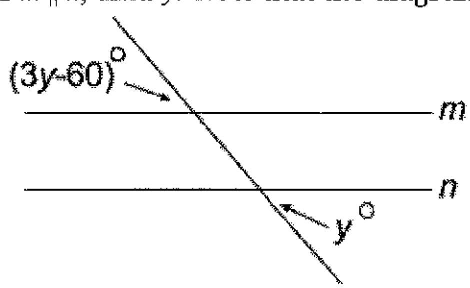
$$\frac{2x^4+38x^3+250x^2+652x+528}{(x+2)(x+4)(x+6)(x+8)}=\frac{2x^4+38x^3+250x^2+668x+608}{(x+2)(x+4)(x+6)(x+8)}$$

$$\frac{-16x}{(x+2)(x+4)(x+6)(x+8)}=\frac{80}{(x+2)(x+4)(x+6)(x+8)}$$

$$-16x=80 \qquad \Rightarrow \qquad x=-5$$

Round 3: Geometry

1. If $m \parallel n$, find y. Note that the diagram is not drawn to scale.

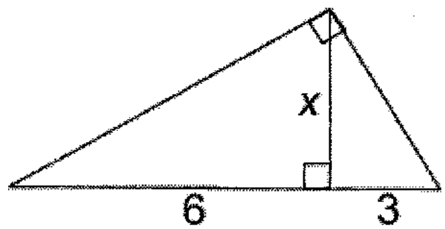


Solution: Since the lines m and n are parallel, we know that the two angles which are labeled in the diagram must be equal since they are alternate exterior angles.

Therefore, we have that

$$3y-60=y$$
$$2y=60$$
$$y=30.$$

2. Find the value of x:



Solution 1 (Right Triangle Altitude Theorem): By the Right Triangle Altitude Theorem, we have that

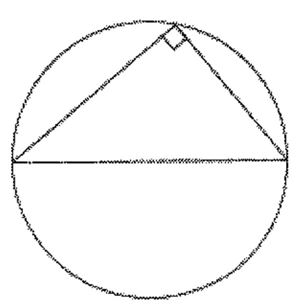
$$\frac{6}{x} = \frac{x}{3}$$
$$x^2 = 18$$
$$x = \sqrt{18} = 3\sqrt{2}$$

Solution 2 (Similarity): We know that the triangle with legs 6 and x is similar to the full triangle since they are both right triangles and share a common angle. Through the same argument, we know that the triangle with legs x and 3 is similar to the full triangle. Hence, we know that the triangle with legs x and 3 is similar to the triangle with legs 6 and x. Therefore, we have that

$$\frac{6}{x} = \frac{x}{3}$$
$$x^2 = 18$$
$$x = \sqrt{18} = 3\sqrt{2}$$

3. A circle passes through all 3 vertices of a triangle with sides 7.5, 10 and 12.5 inches. What is the circumference of the circle?

Solution: Begin by noticing that the triangle described is a scaled 3-4-5 triangle, which we know must be a right triangle. This means that the figure must look like the diagram below.



Since this is a right triangle which is inscribed in a circle, we know by consequence of the inscribed angle theorem that the hypotenuse of the right triangle must be equal to the diameter of the circle.

Therefore, we have that the diameter of this circle must be 12.5 inches, which means that the circumference of the circle is precisely 12.5π inches.

Round 4: Logs, Exponents, and Radicals

1. Solve for x: $log_3 x + log_3(x-6) = 3$

Solution: We have that

$$\begin{aligned}
&log_3 x + log_3(x-6) = 3 \\
&log_3 x(x-6) = 3 \\
&3^3 = x(x-6) \\
&27 = x^2 - 6x \\
&x^2 - 6x - 27 = 0 \\
&(x-9)(x+3) = 0 \\
&x = 9 \text{ or } -3
\end{aligned}$$

However, $log_3(-3)$ is not defined, so we conclude that $x = 9$.

2. Solve for x: $\left(\frac{1}{3}\right)^{-1} \cdot (9)^{x-1} = \left(\frac{1}{9}\right)^{2x-1}$

Solution 1: We have that

$$\begin{aligned}
&\left(\tfrac{1}{3}\right)^{-1} \cdot (9)^{x-1} = \left(\tfrac{1}{9}\right)^{2x-1} \\
&3 \cdot \left(3^2\right)^{x-1} = \left(3^{-2}\right)^{2x-1} \\
&3 \cdot \left(3^{2(x-1)}\right) = 3^{-2(2x-1)} \\
&3^{2x-2+1} = 3^{-2(2x-1)} \\
&3^{2x-1} = 3^{-4x+2} \\
&2x - 1 = -4x + 2 \\
&6x = 3 \\
&x = 0.5
\end{aligned}$$

Solution 2: Begin by noting the following:

$$\left(\tfrac{1}{3}\right)^{-1} = 3 = 9^{1/2}$$
$$\left(\tfrac{1}{9}\right)^{2x-1} = \left(9^{-1}\right)^{2x-1} = 9^{1-2x}$$

Now we have in the original equation that

$$\left(\tfrac{1}{3}\right)^{-1} \cdot (9)^{x-1} = \left(\tfrac{1}{9}\right)^{2x-1}$$

$$9^{1/2} \cdot (9)^{x-1} = 9^{1-2x}$$

$$9^{x-1/2} = 9^{1-2x}$$

Setting the exponents equal we have that

$$x - \tfrac{1}{2} = 1 - 2x$$
$$3x = 1.5$$
$$x = 0.5.$$

3. If $y = log_4 x$, find the real value of x which satisfies $log_y\left(log_4 x^2\right) = 2.$

Solution: We have that $log_4 x^2 = 2log_4 x = 2y$. We plug this into the second equation to see that

$$log_y\left(log_4 x^2\right) = 2$$
$$log_y(2y) = 2$$
$$y^2 = 2y$$
$$y(y-2) = 0 \quad \Rightarrow \quad y = 0 \text{ or } y = 2.$$

However, we know that 0 cannot be the base of a log, so we conclude that $y = 2$. Plugging this information into the first given equation, we have that

$$y = log_4 x$$
$$2 = log_4 x$$
$$4^2 = x$$
$$x = 16.$$

Round 5: Trigonometry

1. If $\tan x + \cot x = \frac{144}{25}$, what is the value of $\frac{1}{\tan x} + \frac{1}{\cot x}$?

Solution: We have that

$$\begin{aligned}
\tan x + \cot x &= \tfrac{144}{25} \\
\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} &= \frac{144}{25} \\
\frac{1}{\frac{\cos x}{\sin x}} + \frac{1}{\frac{\sin x}{\cos x}} &= \frac{144}{25} \\
\frac{1}{\cot x} + \frac{1}{\tan x} &= \frac{144}{25} \\
\frac{1}{\tan x} + \frac{1}{\cot x} &= \frac{144}{25}
\end{aligned}$$

2. Evaluate: $(\tan 10^\circ)(\tan 20^\circ)(\tan 30^\circ)\cdots(\tan 80^\circ)$

Solution: We have that

$$(\tan 10^\circ)(\tan 20^\circ)(\tan 30^\circ)\cdots(\tan 80^\circ)$$
$$\left(\frac{\sin 10^\circ}{\cos 10^\circ}\right)\left(\frac{\sin 20^\circ}{\cos 20^\circ}\right)\left(\frac{\sin 30^\circ}{\cos 30^\circ}\right)\cdots\left(\frac{\sin 80^\circ}{\cos 80^\circ}\right)$$

Recall that $\sin x = \cos(90^\circ - x)$. Using this fact, we have that

$$\left(\frac{\sin 10^\circ}{\cos 10^\circ}\right)\left(\frac{\sin 20^\circ}{\cos 20^\circ}\right)\left(\frac{\sin 30^\circ}{\cos 30^\circ}\right)\cdots\left(\frac{\sin 80^\circ}{\cos 80^\circ}\right)$$
$$\left(\frac{\sin 10^\circ}{\sin 80^\circ}\right)\left(\frac{\sin 20^\circ}{\sin 70^\circ}\right)\left(\frac{\sin 30^\circ}{\sin 60^\circ}\right)\cdots\left(\frac{\sin 80^\circ}{\sin 10^\circ}\right)$$

Every numerator cancels with a denominator, leaving us with a result of 1.

3. Express the following quantity as a single trigonometric function:

$$(sin\ 2x)(csc\ x) - (cos\ 2x)(sec\ x)$$

Solution: We have that

$$(sin\ 2x)(csc\ x) - (cos\ 2x)(sec\ x)$$

$$\frac{2sin(x)cos(x)}{sin\ (x)} - \frac{cos^2\ (x) - sin^2\ (x)}{cos\ (x)}$$

$$2cos(x) - \frac{cos^2\ (x) - sin^2\ (x)}{cos\ (x)}$$

$$\frac{2cos^2(x)}{cos(x)} - \frac{cos^2\ (x) - sin^2\ (x)}{cos\ (x)}$$

$$\frac{cos^2\ (x) + sin^2\ (x)}{cos\ (x)} = \frac{1}{cos(x)} = sec\ (x)$$

Team Round

1. Determine the digit A such that the six digit number $3730A5$ is divisible by 21.

Solution 1: Since $21 = 3 \times 7$, we know that if $3730A5$ is divisible by 21 then it must also be divisible by 3. We know that a number is divisible by 3 if its digits add up to a number which is divisible by 3. This means that $3+7+3+0+A+5 = 18+A$ must be divisible by 3. Therefore, we can narrow down that A must be equal to 0, 3, 6 or 9.

Likewise, we also know that 7 must divide the number. Dividing 7 into 373000 leaves a remainder of 5. This means that 7 must divide $5+A5$. Of the four possibilities for A which we determined above, the only option which makes $5+A5$ divisible by 7 is $A=6$.

Solution 2: Use the same argument as in the first paragraph of solution 1 to conclude that A is 0, 3, 6 or 9. Moreover, we know that $3730A5$ must also be divisible by 7. Dividing $3730A5$ by 7 leaves us with 7 dividing into 6A5. Try each possible value for A to see if 6A5 is divisible by 7:

605/7 = 86 Remainder 3
635/7 = 90 Remainder 5
665/7 = 95
695/7 = 99 Remainder 2

We conclude that A = 6.

2. The period of a wave is the reciprocal of the wave's frequency, f. A test device emits sound waves which are linearly increasing in frequency over time. At time $t = 0$ sec, $f = 240$ Hz. At $t = 2$ min 40 sec, $f = 400$ Hz. What is the period of the sound waves emitted by the device at $t = 1$ min 25 sec?

Solution: First, we need to determine the function which describes the behavior of the device. We are given that the sound waves are linearly increasing in frequency over time, which implies that $f(t) = pt + q$, where t represents time in seconds and p and q are some fixed numbers.

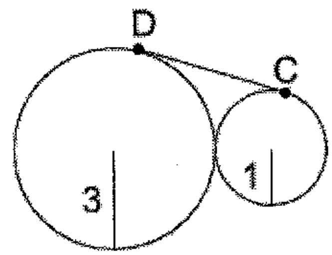
Next, we are given two data points: $f(0) = 240$ and $f(160) = 400$. The slope of the function p must be such that

$$p = \frac{400-240}{160-0} = \frac{160}{160} = 1.$$

Moreover, since we know that $f(0) = p(0) + q = 240$, we have that $q = 240$.

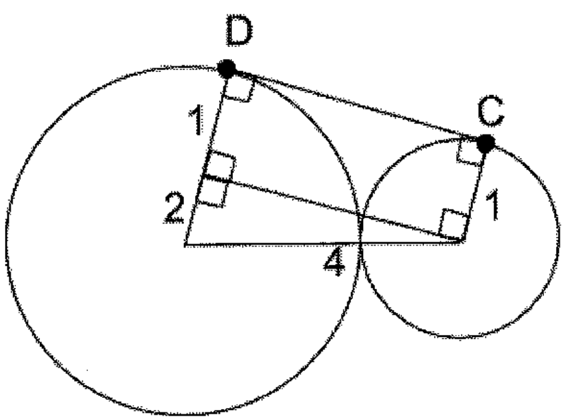
Now we can determine the frequency of the sound waves emitted ar 1 min 25 sec (or 85 sec) by plugging in $f(85) = 1 \times (85) + 240 = 325$ Hz. Finally, since the period of a wave is equal to the reciprocal of the wave's frequency, we have that the desired period is $\frac{1}{325}$ seconds.

3. If the two circles in the diagram are tangent to one another, find the length of the external tangent segment DC.



Solution:

With the given information, we can draw in some details below:



Notice that the length of DC is also the length of the leg of the triangle with hypotenuse 4 and second leg 2. We can use the Pythagorean Theorem to get that:

$$DC^2 + 2^2 = 16$$
$$DC^2 = 12$$
$$DC = \sqrt{12} = 2\sqrt{3}.$$

4. Solve for x: $216^{\frac{x-3}{3}} = 36^{\frac{2}{x}}$.

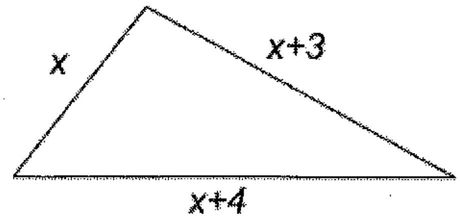
Solution: We have that

$$216^{\frac{x-3}{3}} = 36^{\frac{2}{x}}$$
$$\left(6^3\right)^{\frac{x-3}{3}} = \left(6^2\right)^{\frac{2}{x}}$$
$$6^{x-3} = 6^{\frac{4}{x}}$$

Therefore, we have that

$$x - 3 = \frac{4}{x}$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0 \quad \Rightarrow \quad x = -1 \text{ or } x = 4.$$

5. Find $x > 0$ if the cosine of the smallest angle of this triangle is $\frac{5}{6}$.



Solution: The smallest angle of the triangle is the one opposite from the side x. Call this angle C. From the law of cosines, we have that

$$x^2 = (x+3)^2 + (x+4)^2 - 2(x+3)(x+4)cos(C)$$

Since we know that $cos(C) = \frac{5}{6}$, we now have

$$x^2 = (x+3)^2 + (x+4)^2 - 2(x+3)(x+4)\tfrac{5}{6}$$
$$x^2 = (x^2+6x+9) + (x^2+8x+16) - 2(x^2+7x+12)\tfrac{5}{6}$$
$$x^2 = x^2+6x+9+x^2+8x+16-\tfrac{5}{3}x^2-\tfrac{35}{3}x-20$$
$$0 = -\tfrac{2}{3}x^2+\tfrac{7}{3}x+5$$
$$2x^2-7x-15=0$$
$$(2x+3)(x-5)=0$$

Since $x > 0$, we have that $x = 5$.

6. If $log_2\left(log_3\left(log_x(3)\right)\right) = 1$ what is $log_3 x$?

Solution: We have that

$$log_2\left(log_3\left(log_x(3)\right)\right) = 1$$
$$log_3\left(log_x(3)\right) = 2^1 = 2$$
$$log_x(3) = 3^2 = 9$$
$$3 = x^9$$
$$log_3 3 = log_3 x^9$$
$$1 = 9 log_3 x$$
$$log_3 x = \tfrac{1}{9}$$

7. A fair coin is flipped 5 times. What is the probability that at least 2 heads are flipped?

Solution 1: We know that the desired probability is precisely equal to

1 - Prob(No heads are flipped) - Prob(Exactly one head is flipped)

This is equal to

$$1-\left(\tfrac{1}{2}\right)^5-(5\ choose\ 1)\times\tfrac{1}{2}\times\left(\tfrac{1}{2}\right)^4$$
$$1-\left(\tfrac{1}{2}\right)^5-5\left(\tfrac{1}{2}\right)^5=1-6\left(\tfrac{1}{2}\right)^5=\tfrac{13}{16}$$

Solution 2: To determine the desired probability of at least two heads occurring, we will compute the probability of the event's complement and then subtract that from 1.

In this case, the complement is when either no heads are flipped or when exactly one head is flipped.

There is only one arrangement with no heads: TTTTT
There are five arrangements with one head: HTTTT, THTTT, TTHTT, TTTHT, TTTTH

In total, there are 32 possible arrangements after flipping a coin five times, since 32 = 2^5.
Since the complement of the desired event occurs 6 out of 32 times, we know that the desired event occurs 26 out of 32 times, leaving a probability of $\frac{13}{16}$.

8. What is the sum of all the coefficients in the expansion of $(23x-16y)^{10}$?

Solution: To determine the sum of the coefficients in the expansion, simply set both x and y to be equal to 1 and evaluate. This gives $(23(1)-16(1))^{10}=7^{10}$.

9. Suppose there is a function f such that $\frac{f(x^2+1)}{x} = k$ for some real number k and for all $x \neq 0$. If $m \neq 0$, what is $\frac{f\left(\frac{m^2+16}{16}\right)}{m}$ in terms of k?

Solution: We have that

$$\frac{f\left(\frac{m^2+16}{16}\right)}{m} = \frac{f\left(\frac{m^2}{16}+1\right)}{m} = \frac{f\left(\left(\frac{m}{4}\right)^2+1\right)}{m} = \frac{1}{4} \times \frac{f\left(\left(\frac{m}{4}\right)^2+1\right)}{\frac{m}{4}} = \frac{1}{4}k$$