WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014
Round 1: Algebraic Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. A manager earns a base salary of $1000 per month plus a 5% commission on all sales she makes. Her goal is to earn $3200 per month total. What dollar value of sales must she make each month in order to meet her goal?

2. The perimeter of a rectangular playground is 82 ft. If the length of the playground is 5 more than 2 times its width, what is the area of the playground?

3. Paul has $60 more than double Shirley's amount. Matilda has triple Paul's amount. If Paul gave $250 to Shirley and if Matilda gave $500 to Paul, then Paul would have $30 more than Matilda. How much money did Paul start with?

ANSWERS

(1 pt.) 1. $ ________________________________

(2 pts.) 2. ________________________________ Square Feet

(3 pts.) 3. $________________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014
Round 2: Number Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Express the following sum in base 10: $2103_4 + 1101_2$

2. What is the least integer greater than 1 that is a square, cube, fifth power and sixth power?

3. An infinitely long string of digits begins with a 6 and has the property that any two consecutive digits form a number that is divisible by either 17 or 23. What is the 2014th digit?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014

Round 3: Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Simplify: $\dfrac{2}{1-\frac{1}{1+\frac{2}{3}}}$

2. After a productive October, Henry received a 7% raise as a reward. Now Henry earns a daily wage of $87.74. What was Henry's daily wage before his raise?

3. Kayla held equal amounts of money in two savings accounts, one which pays 6% simple interest per year and one which pays 8% simple interest per year. If after one year Kayla earned $260 more in interest from the 8% account than from the 6% account, how much money did she initially put in each account?

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. $______________________________

(3 pts.) 3. $ ______________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014
Round 4: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. At a class party, 6 people ate hamburgers, 8 people ate hotdogs, and 4 people ate both hotdogs and hamburgers. If 5 people ate nothing, how many total people were at the party?

2. How many subsets of S exist, given that S = {all primes between 0 and 16}?

3. Suppose set A has 128 subsets, set B has 64 subsets, and set C has 32 subsets. If $A \cup B$ has 9 elements and $A \cup C$ has 12 elements, what is the number of elements in the set $(A \cap B) \cup C$?

ANSWERS

(1 pt.) 1. ______________________ people

(2 pts.) 2. ______________________ subsets

(3 pts.) 3. ______________________ elements

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014
Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

APPROVED CALCULATORS ALLOWED

1. Greg wanted to buy a new tablet, but its price of \$250.88 was too much. He watched the price of the tablet closely, noting that it dropped to \$188.16, then to \$141.12, and finally to \$105.84. Greg bought the tablet after its price had been reduced again, at a rate consistent with the earlier price reductions. How much did he pay for the tablet?

2. If m and n are whole numbers such that $1 < n < 101$ and $50 < m < 101$, then what is the greatest possible value of the expression $\frac{mn^3+m^2n}{mn^4}$?

3. Suppose that a painting job can be done by 9 Red Team painters in 8 hours or by 16 Blue Team painters in 9 hours. If 4 Red Team painters and 4 Blue Team painters begin a new job at 7:00 A.M. and 30 minutes is allotted for lunch, at what time will they complete the job?

4. The larger of 2 numbers is 4 less than 3 times the smaller number. If 3 more than twice the larger number is decreased by twice the smaller number, the result is 11. Find the larger number.

5. What is the smallest positive integer whose digits consist entirely of 2s and 3s, which has at least one 2 and one 3 among its digits, and which is divisible by both 2 and 3?

6. Let S = {1, 2, …, 10}. Two different elements of S are selected, one to be used as a numerator and one to be used as a denominator of a number. What percent of all possible fractions formed this way are integers?

7. For a set S, let S’ denote its complement. Now suppose that A = {0, 1, 2, 3, 4}, B ={odd integers}, and C = {even strictly positive integers}. If the universal set is given by the set of all integers, what is $(A \cup B') \cap (C' \cap A')$?

8. A blacksmith melts 3 metal cubes of 3 cm, 4 cm, and 5 cm sides and uses all the melted material to form one large cube. What is the ratio of the total surface area of the smaller cubes to the surface area of the new large cube?

Freshman Meet 2 - December 10, 2014
Team Round Answer Sheet

1. $________________________

2. ________________________

3. ________________________

4. ________________________

5. ________________________

6. ________________________ %

7. ________________________

8. ________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014 ANSWER KEY

Round 1:

1. \$44,000 (Shrewsbury)
2. 348 Sq. Feet (Auburn)
3. \$360 (Assabet Valley)

Round 2:

1. 160 (Bartlett)
2. 2^{30} or 1,073,741,824 (Worcester Academy)
3. 3 (AMSA)

Round 3:

1. 5 (Leicester)
2. \$82.00 or \$82 (Bartlett)
3. \$13000 (Assabet Valley)

Round 4:

1. 15 (Wachusett)
2. 64 (Southbridge)
3. 9 (Algonquin)

TEAM Round

1. \$79.38 (Millbury)
2. 13 (Shepherd Hill)
3. 7:30 P.M. or 19:30 (St. John's)
4. 8 (Shepherd Hill)
5. 2232 (AMSA)
6. $\frac{170}{9}\%$ or 18.889% or $18.\overline{8}\%$ (Worcester Academy)
7. {..., -6, -4, -2} or {strictly negative even numbers} (Quaboag)
8. $\frac{25}{18}$ or 25:18 (West Boylston)

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014 SOLUTIONS

Round 1: Algebraic Word Problems

1. A manager earns a base salary of $1000 per month plus a 5% commission on all sales she makes. Her goal is to earn $3200 per month total. What dollar value of sales must she make each month in order to meet her goal?

SOLUTION: Let x denote the dollar value of sales that the manager must make in order to accomplish her goal. Then the problem simplifies to

$$(0.05)x + 1000 = 3200$$
$$(0.05)x = 2200$$
$$x = \frac{2200}{0.05}$$
$$x = \frac{2200}{\frac{5}{100}}$$
$$x = 2200 \times \frac{100}{5}$$
$$x = 2200 \times 20$$
$$x = 44{,}000$$

2. The perimeter of a rectangular playground is 82 ft. If the length of the playground is 5 feet longer than 2 times its width, what is the area of the playground?

SOLUTION: Let x denote the width of the playground. Then we have that the length of the playground is given by 2x + 5. Now we use the information about the perimeter to solve for x:

$$82 = x + x + (2x+5) + (2x+5)$$
$$82 = 6x + 10$$
$$6x = 72$$
$$x = 12$$

Now that we know x, we have that the width is 12 ft and the length is 29 ft. To compute the area, we multiply the width by the length to get an answer of 348 Sq. ft.

3. Paul has $60 more than double Shirley's amount. Matilda has triple Paul's amount. If Paul gave $250 to Shirley and if Matilda gave $500 to Paul, then Paul would have $30 more than Matilda. How much money did Paul start with?

SOLUTION: Let x denote Shirley's amount at the beginning. Then we have that

	Start	Finally
Paul:	2x + 60	2x + 60 - 250 + 500 = 2x + 310
Shirley:	x	x + 250
Matilda:	6x + 180	6x + 180 - 500 = 6x - 320

We are given that Paul ends with $30 more than Matilda, which we can write as:

$$2x + 310 = 6x - 320 + 30$$
$$600 = 4x$$
$$x = 150$$

Therefore, Paul started with 2(150) + 60, or 360 dollars.

Round 2: Number Theory

1. Express the following sum in base 10: $2103_4 + 1101_2$

SOLUTION: We begin by writing both numbers in base 10:

$$2103_4 = 2(4^3) + 4^2 + 3 =$$
$$2 \times 64 + 16 + 3 =$$
$$128 + 19 = 147$$

$$1101_2 = 2^3 + 2^2 + 1 =$$
$$8 + 4 + 1 = 13$$

Now we have that the sum is equal to 147 + 13 = 160.

2. What is the least integer greater than 1 that is a square, cube, fifth power and sixth power?

Solution: Let x be the number we are looking for. By the fundamental theorem of arithmetic, we know that x has a unique prime factorization. That is, we can uniquely express x as the product of prime integers, where each prime factor is raised to an integer power greater than or equal to one.

If x is a sixth power of some integer, then we know that each of x's prime factors must have a power that is a multiple of 6. Otherwise taking the sixth root of x would yield some prime factors that have non-integer powers, which cannot be the case.

By the same logic, we know that if x is a fifth power, then it must be the case that each of x's prime factors is raised to a power that is a multiple of five. Since the lcm of 5 and 6 is 30, we conclude that each of x's prime factors must be raised to a power that is a multiple of 30.

The smallest such prime factorization that is greater than 1 is given simply by 2^{30}. Notice that this number is also a square ($2^{30} = 2^{15} \times 2^{15}$) and a cube ($2^{30} = 2^{10} \times 2^{10} \times 2^{10}$)

Hence the solution is x = 2^{30}.

3. An infinitely long string of digits begins with a 6 and has the property that any two consecutive digits form a number that is divisible by either 17 or 23. What is the 2014th digit?

SOLUTION: Begin by noting that the set of two-digit multiples of 17 is given by {17, 34, 51, 68, 85} and the set of two-digit multiples of 23 is given by {23, 46, 69, 92}. Therefore, the string must begin with either 68 or 69.

In the first case, the string must proceed 68517 but then there is no possible two digit number that can follow starting with a 7, so this case is ruled out.
Therefore, the string begins 68234692… with a pattern that repeats every five digits. Therefore, the 2014th digit is the same as the fourth digit in this pattern, which is 3.

Round 3: Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

1. Simplify: $\dfrac{2}{1-\frac{1}{1+\frac{2}{3}}}$

Solution:

$$\frac{2}{1-\frac{1}{1+\frac{2}{3}}}=$$

$$\frac{2}{1-\frac{1}{\frac{5}{3}}}=$$

$$\frac{2}{1-\frac{3}{5}}=$$

$$\frac{2}{\frac{2}{5}}=$$

$$2\times\frac{5}{2}=5$$

2. After a productive October, Henry received a 7% raise as a reward. Now Henry earns a daily wage of $87.74. What was Henry's daily wage before his raise?

Solution: Let x be Henry's pre-raise wage. Then we solve the equation

$$1.07x=87.74$$
$$x=\frac{87.74}{1.07}$$
$$x=\frac{8774}{107}$$
$$x=82.00$$

3. Kayla held equal amounts of money in two savings accounts, one which pays 6% simple interest per year and one which pays 8% simple interest per year. If after one year Kayla earned $260 more in interest from the 8% account than from the 6% account, how much money did she initially put in each account?

Solution: Let x denote the amount of money Kayla initially put in each account. Then we have that:

$$.06x + 260 = 0.08x$$
$$0.02x = 260$$
$$2x = 26000$$
$$x = 13000$$

Round 4: Set Theory

1. At a class party, 6 people ate hamburgers, 8 people ate hotdogs, and 4 people ate both hotdogs and hamburgers. If 5 people did not eat at the party, how many people are at the party?

Solution: The total number of people at the party is the sum of the # of people who ate nothing, the # of people who ate only hotdogs, the # of people who ate only hamburgers, and the # of people who ate both hotdogs and hamburgers. (These four groups are mutually exclusive and exhaustive)

Since 4 people ate both hotdogs and hamburgers and 6 people ate hamburgers, we conclude that 2 people ate only hamburgers. Likewise, since 8 people ate hotdogs, we conclude that 4 people ate only hotdogs. Therefore there are 5 + 4 + 2 + 4 = 15 people at the party.

2. How many subsets of S exist, given that S = {all primes between 0 and 16}?

Solution: We have that S = {2, 3, 5, 7, 11, 13}. Since the cardinality of S is 6, we know that S has a total of $2^6 = 64$ subsets.

3. Suppose set A has 128 subsets and set B has 64 subsets and set C has 32 subsets. If $A \cup B$ has 9 elements and $A \cup C$ has 12 elements, what is the number of elements in the set $(A \cap B) \cup C$?

Solution: From the information about the subsets, we know that A has 7 elements, B has 6 elements, and C has 5 elements. Given this and the fact that $A \cup B$ has 9 elements, we know by the identity $|A \cup B| + |A \cap B| = |A| + |B|$ that $A \cap B$ must have 4 elements. Moreover, we can use the same logic to determine that $A \cap C$ is the empty set.

Since every element of $A \cap B$ is a member of set A and $|A \cap C| = 0$, we have that the cardinality of set $(A \cap B) \cup C$ is given by $|A \cap B| + |C| = 4 + 5 = 9$.

Team Round

1. Greg wanted to buy a new tablet, but its price of $250.88 was too much. He watched the price of the tablet closely, noting that it dropped to $188.16, then to $141.12, and finally to $105.84. Greg bought the tablet after its price had been reduced again, at a rate consistent with the earlier price reductions. How much did he pay for the tablet?

Solution: The first sale reduced the price by 25%, since

$$\frac{250.88-188.16}{250.88}=0.25$$

The second price reduction reduced the price of 188.16 by 25%, since

$$\frac{188.16-141.12}{188.16}=0.25$$

Therefore, if we reduce the price of 105.84 by 25%, we will have the answer:

$$0.75\times 105.84=79.38$$

Greg paid $79.38 for the tablet.

2. If m and n are whole numbers such that 1 < n < 101 and 50 < m < 101, then what is the greatest possible value of the expression $\frac{mn^3+m^2n}{mn^4}$?

Solution: First simplify the expression:

$$\frac{mn^3+m^2n}{mn^4}=\frac{1}{n}+\frac{m}{n^3}$$

Since n is always in the denominator, we need to make it as small as possible. Likewise, since m only appears in the numerator we need to make it as large as possible. This gives that n = 2 and m = 100, which means that the maximum value of the expression is given by

$$\frac{1}{2}+\frac{100}{8}=0.5+12.5=13.$$

3. Suppose that a painting job can be done by 9 Red Team painters in 8 hours or by 16 Blue Team painters in 9 hours. If 4 Red Team painters and 4 Blue Team painters begin a new job at 7:00 A.M. and 30 minutes is allotted for lunch, at what time will they complete the job?

Solution: It takes (9 Red Teamers) x (8 hours) to complete a job, or in other words 72 Red-Team-hours. It takes (16 Blue Teamers) x (9 hours) to complete a job, or 144

Blue-Team-hours. Given this information we conclude that 1 Red-Team-hour is equivalent to 2 Blue-Team-hours.

This means that 4 Red Team painters and 4 Blue Team painters work at the same rate as 6 Red Team Painters. If it takes 72 Red-Team-hours to complete the job, then 6 Red Team painters can finish it in 12 hours. Adding the half hour for lunch, we have that the group of 4 Red Team painters and 4 Blue Team painters will finish the job 12.5 hours after beginning, at the time of 7:30 P.M.

4. The larger of two numbers is 4 less than 3 times the smaller number. If 3 more than twice the larger number is decreased by twice the smaller number, the result is 11. Find the larger number.

Solution: Let y denote the large number and x denote the small number. Then we have that y = 3x - 4 and that 2y + 3 - 2x = 11. Substituting the first equation into the second, we have that

$$2(3x-4)+3-2x=11$$
$$6x-8+3-2x=11$$
$$4x=16$$
$$x=4$$

Therefore, y = 3(4) - 4 = 8.

5. What is the smallest positive integer whose digits consist entirely of 2s and 3s, which has at least one 2 and one 3 among its digits, and which is divisible by both 2 and 3?

Solution: Let n be the integer, let k be the number of digits in the n, and let s be the sum of the digits in n.

If n is divisible by 2, then its last digit must be 2. If n is divisible by 3, then s must be a multiple of 3.

Now we know k = 1 does not work since 2 is not divisible by 3 (and also does not contain at least one digit equal to 3). Likewise k = 2 does not work since 32 is not divisible by 3. For k = 3, either either n has two 2s and one 3 or it has two 3s and one 2. But in neither case is s divisible by three, so neither will work.

Finally at k = 4, we see that the number 2232 satisfies all necessary criteria.

6. Let S = {1, 2, …, 10}. Two different elements of S are selected, one to be used as a numerator and one to be used as a denominator of a number. What percent of all possible fractions formed this way are integers?

Solution: Note that two different elements are chosen so numbers like 4/4 are not possible. Now, for each possible numerator, we can determine how many possible denominators provide an integer result.

Numerator	Denominators yielding Integer Results
1	(none)
2	1
3	1
4	1, 2
5	1
6	1, 2, 3
7	1
8	1, 2, 4
9	1, 3
10	1, 2, 5

There are 17 total denominators in the list above yielding integer results. The total number of possible fractions created according to the procedure described is equivalent to the number of ways we can pick two objects from a set of 10 objects. This is given by 10 x 9 = 90.

Therefore, the percentage of all possible fractions formed according to the procedure that are integers is given by

$$\frac{17}{90}\times 100\,\% = \frac{170}{9}\,\% \text{ or } 18.889\,\%$$

7. For a set S, let S’ denote its complement. Now suppose that A = {0, 1, 2, 3, 4}, B ={odd integers}, and C = {even strictly positive integers}. If the universal set is given by the set of all integers, what is $(A \cup B') \cap (C' \cap A')$?

Solution: We start by computing $A \cup B'$. Since the universal set is the set of all integers, B’ is given by {even integers}. Therefore $A \cup B'$ is {even integers} $\cup$ {1, 3}.

Now we compute $C' \cap A'$. The set C’ is given by {all negative integers} $\cup$ {0} $\cup$ {positive odd integers}. The set A’ is given by {all integers except 0, 1, 2, 3 , or 4}. Therefore, the set $C' \cap A'$ is given by {..., -3, -2, -1} $\cup$ {5, 7, 9, …}.

Now putting it all together, we see that $(A \cup B') \cap (C' \cap A')$ = {..., -6, -4, -2}.

8. A blacksmith melts 3 metal cubes of 3 cm, 4 cm, and 5 cm sides and uses all the melted material to form one large cube. What is the ratio of the total surface area of the smaller cubes to the surface area of the new large cube?

Solution: The volume of the new large cube is given by $3^3+4^3+5^3=216\,\text{cm}^3$. This implies that the side of the large cube is given by $\sqrt[3]{216}=6\text{ cm}$.

Therefore, the ratio of the total surface area of the small cubes to the surface area if the large cube is given by

$$\frac{3^2\times6+4^2\times6+5^2\times6}{6^2\times6}=$$

$$\frac{6\times(9+16+25)}{6\times36}=$$

$$\frac{50}{36}=\frac{25}{18}$$