Freshman Meet 2 – January 9, 2013
Round 1: Algebraic Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. What number is doubled when two-thirds of it is added to 36?

2. Eric starts walking to school at 7:00 AM. If he walks at a rate of 4 miles per hour, he will be five minutes late; if he walks at a rate of 5 miles per hour, he will be ten minutes early. At what time does school begin? Indicate AM or PM.

3. Tom, Huck, and Finn are painting a fence and complete the job in 1 hour. Working together, Tom and Finn would need 1 hour and 40 minutes, while Huck and Finn working together would need 1 hour and $21\frac{9}{11}$ minutes. How long, _in hours_, would it take Finn to paint the fence working alone?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________ [indicate AM or PM]

(3 pts.) 3. ______________________ hours

Freshman Meet 2 – January 9, 2013
Round 2: Number Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

Note: a subscript indicates that number's base.

1. The greatest common divisor (also called "greatest common factor") of two or more numbers is the largest positive integer that divides all of the numbers without leaving a remainder. Find the gcd of 693 and 882.

2. Find the integer between 300 and 400 that is divisible by 2, 3, 4, 6, 8, and 9.

3. In what base does the numeral 33 equal 24_7?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Freshman Meet 2 – January 9, 2013
Round 3: Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. What single percentage discount is equal to two successive discounts of 20% and 10%?

2. If $1.4x = 2.1y$, $1.2y = 2.5z$, and $x = kz$, find k.

3. Evaluate and express as a simplified fraction:

$$\frac{3.\overline{45} \times 1.8\overline{3} \div 2.\overline{6}}{1.1\overline{6} \times 2.374\overline{9}}.$$

The lines above the digits denote repeating decimals.

ANSWERS

(1 pt.) 1. ______________________ %

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Freshman Meet 2 – January 9, 2013
Round 4: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Given $A = \{0, 4, 8, 12\}$, $B = \{5, 6, 7\}$, and $C = \{2, 4, 6, 8\}$, find $(A \cap C) \cup B$.

2. Of the 75 students surveyed, 46 own a dog and 33 own a cat. How many students own neither if 12 students own both?

3. One hundred math students were surveyed about their preferred subjects in school: algebra, geometry, and number theory. Sixty-one liked algebra, 42 liked geometry, and 43 liked number theory. Eight liked all three, 18 liked algebra and number theory but not geometry, and 10 liked algebra and geometry but not number theory. Every student liked at least one of these subjects. Given this information, if a student who likes both geometry and number theory is chosen at random, what is the probability that she also likes algebra? Express your answer as a fraction.

ANSWERS

(1 pt.) 1. { ______________________ }

(2 pts.) 2. ______________________ students

(3 pts.) 3. ______________________

Freshman Meet 2 – January 9, 2013
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. The 17th century French amateur mathematician Pierre de Fermat proposed the theorem that any prime that leaves a remainder of 1 upon division by 4 can be written as the sum of two squares. Find the two perfect squares that sum to 113. Express your answer as an ordered pair with the smaller square first.

2. At a recent County Fair, admission for adults was \$4, senior citizens were charged \$3, and children's tickets cost \$2. If 8000 people attended and total gate receipts were \$23150, how many children attended if the ratio of children to adults (not including senior citizens) was $3 : 2$?

3. Given that $f(x) = x^3 - 6x + 11$ and $g(x) = 3x^2 - 6$, evaluate $f(x)$ at the smallest value of x for which $g(x) = 0$.

4. What is the remainder when the product of all the prime numbers between 1 and 100 is divided by 4? Express the remainder as a whole number.

5. Sam bought some glasses at \$5 each. Five were broken, but he sold the rest at \$8 each and made a profit of 20% on the deal. How many glasses did he buy?

6. Let R, S, and T be subsets of the universal set $U = \{1, 2, 3, 4, 5, 6\}$. Given that $R = \{3, 5\}$, $S \cap T = \{3\}$, $S \cup T = \{1, 2, 3, 4, 6\}$, and $R \cup T = \{1, 3, 4, 5\}$, find S.

7. At 50 mph Henry can drive 300 miles on 10 gallons of gasoline. At 55 mph he gets 3 more miles per gallon than at 50 mph. At 55 mph, how many hours will it take him to burn 25 gallons of gas?

8. Given that

$$22! = 1124000727777ab7680000$$

where a and b are base 10 digits, find the ordered pair (a, b).

Freshman Meet 2 – January 9, 2013
TEAM ROUND ANSWER SHEET

1. (________ , ________)

2. ________________ children

3. ________________

4. ________________

5. ________________ glasses

6. { ________________ }

7. ________________ hours

8. (________ , ________)

Freshman Meet 2 – January 9, 2013
ANSWERS

ROUND 1
(Algonquin, Worcester Academy, Westborough)

1. 27
2. 8:10 AM (must have AM)
3. 3 hours

ROUND 2
(Doherty, Quaboag, Doherty)

1. 63
2. 360
3. 5

ROUND 3
(Leicester, Worcester Academy, Assabet Valley)

1. 28%
2. $25/8 = 3\frac{1}{8} = 3.125$
3. $6/7$

ROUND 4
(Holy Name, Bartlett, QSC)

1. $\{4, 5, 6, 7, 8\}$ (need all, any order)
2. 8 students
3. $4/5$

TEAM ROUND
(QSC, Assabet Valley, QSC, Tahanto, Southbridge, Westborough, Quaboag, QSC)

1. $(49, 64)$
2. 2550 children
3. $11 + 4\sqrt{2}$ or $4\sqrt{2} + 11$
4. 2
5. 20 glasses
6. $\{2, 3, 6\}$ (need all three, any order)
7. 15 hours
8. $(6, 0)$

Freshman Meet 2 – January 9, 2013
FULL SOLUTIONS

ROUND 1

1. We have $36 + \frac{2}{3}x = 2x$, or $x = \boxed{27}$.

2. Multiply through by 60 minutes/hour and let x be the number of minutes after 7:00 AM that school starts. Then,
$$4(x + 5) = 5(x - 10),$$
and $x = 70$. Therefore, school starts at $\boxed{8{:}10 \text{ AM}}$.

3. In general, for this type of problem, if two people would finish a job in a and b hours, then the total time t (in hours) they would take working together can be solved for as follows:
$$\begin{aligned} \frac{t}{a} + \frac{t}{b} &= 1 \\ \frac{1}{a} + \frac{1}{b} &= \frac{1}{t} \\ \left[\frac{1}{a} + \frac{1}{b}\right]^{-1} &= t \end{aligned}$$
It takes Tom and Finn 1 hour and 40 minutes, or 5/3 hours, to paint the fence, so it would take Huck $\left[\frac{1}{1} - \frac{1}{5/3}\right]^{-1} = \frac{5}{2}$ hours working alone. We are also given that it takes Huck and Finn 1 hour and $21\frac{9}{11}$ minutes, or 15/11 hours, working together.
Therefore, it takes Finn $\left[\frac{1}{15/11} - \frac{1}{5/2}\right]^{-1} = \boxed{3}$ hours working alone.

ROUND 2

1. **METHOD I:** By prime factorization, $693 = 3^2 \cdot 7 \cdot 11$ and $882 = 2 \cdot 3^2 \cdot 7^2$, so the gcd is $3^2 \cdot 7 = \boxed{63}$.

   METHOD II: To find the gcd of two numbers, especially if they are large, the preferred method is the EUCLIDEAN ALGORITHM, first described by Euclid in 300 BC. In the Euclidean algorithm, the smaller number is divided into the larger number and the integer remainder taken. This process is repeated, and the last nonzero number is the gcd. For this problem:

$$\begin{aligned} 882 &= 1 \times 693 + 189 \\ 693 &= 3 \times 189 + 126 \\ 189 &= 1 \times 126 + \boxed{63} \\ 126 &= 2 \times 63 + 0 \end{aligned}$$

The advantage to this method is that the two numbers need not be factored. For large numbers (≈100 base 10 digits), prime factorization in a reasonable amount of time is not possible even with supercomputers, but the number of steps required in the Euclidean algorithm is at most 5 times the number of base 10 digits in the smaller number.

2. If a number is divisible by 2, 3, 4, 6, 8, and 9, it is also divisible by their lcm, which is 72. The only number between 300 and 400 that is divisible by 72 is $\boxed{360}$.

3. We have $24_7 = 14_{10} + 4 = 18_{10}$. Also, $33_n = 3n + 3$, so the base we need is $n = \boxed{5}$.

ROUND 3

1. Compute: $0.8 \times 0.9 = 0.72$, so the discount is equal to $1 - 0.72 = \boxed{28\%}$.

2. We have $x = \frac{3}{2}y$ and $y = \frac{25}{12}z$, so $x = \frac{3}{2} \cdot \frac{25}{12}z$. Simplifying, $\frac{3}{2} \cdot \frac{25}{12} = \boxed{\frac{25}{8}}$.

3. First, convert all of the repeating decimals to fractions:

$$\frac{3.\overline{45} \times 1.8\overline{3} \div 2.\overline{6}}{1.1\overline{6} \times 2.374\overline{9}} = \frac{(38/11) \times (11/6) \div (8/3)}{(7/6) \times (19/8)}.$$

Then, multiply all of the fractions together, inverting as necessary for division. Most of the terms cancel.

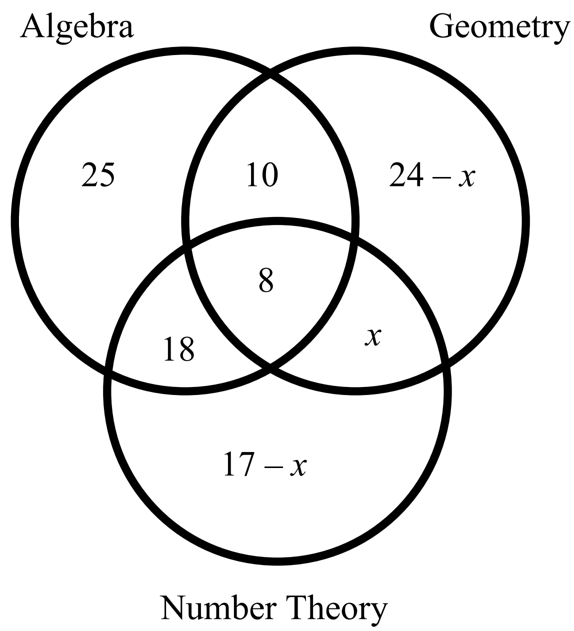
$$\frac{38}{11} \times \frac{11}{6} \times \frac{3}{8} \times \frac{6}{7} \times \frac{8}{19} = \boxed{\frac{6}{7}}.$$

ROUND 4

1. First find that $A \cap C = \{4, 8\}$. Then, take the union with $B = \{5, 6, 7\}$ to find that $(A \cap C) \cup B = \boxed{\{4, 5, 6, 7, 8\}}$.

2. Using the Principle of Inclusion-Exclusion, we find that $46 + 33 - 12 = 67$ students have either a dog or a cat. Therefore, the remaining $75 - 67 = \boxed{8}$ students have neither.

3. Fill in a Venn diagram with the given information:



We are not given the number of students who liked geometry and number theory but not algebra, so call that number x. Since a total of 100 students were surveyed, we then have $25 + 10 + 8 + 18 + (24 - x) + x + (17 - x) = 100$, or $x = 2$. This means that a total of 10 students like geometry and number theory, and 8 of those students also like algebra. The required probability is therefore $8/10 = \boxed{4/5}$.

TEAM ROUND

1. Use trial and error: $113 - 100 = 13$ is not a square, $113 - 81 = 32$ is not a square, but $113 - 64 = 49$ is. The theorem guarantees existence, but does not give any information on how to find the two squares. Answer: $\boxed{(49, 64)}$

2. To satisfy the ratio, let there be $3x$ children and $2x$ adults. That leaves the remaining $8000 - 5x$ people to be senior citizens. Therefore,

$$\begin{aligned} 2(3x) + 4(2x) + 3(8000 - 5x) &= 23150 \\ 14x + 24000 - 15x &= 23150 \\ 850 &= x \end{aligned}$$

The number of children was $3x$, so $3(850) = \boxed{2550}$.

3. Solving $g(x) = 0$ gives $x = \pm\sqrt{2}$. We want the smaller value, so evaluating f at $-\sqrt{2}$, we have that $f(-\sqrt{2}) = -2\sqrt{2} + 6\sqrt{2} + 11 = \boxed{11 + 4\sqrt{2}}$.

4. **METHOD I:** 2 is prime, so the product must be even. Also, 2 is the only even prime, so the product cannot be divisible by 4. Therefore, the remainder upon division by 4 must be $\boxed{2}$.

   METHOD II: Using modular arithmetic, $2 \equiv 2 \pmod 4$. All other primes are odd, so they are equivalent to $\pm 1 \pmod 4$. Therefore, the product is equivalent to $\pm 2 \pmod 4$, which is just $\boxed{2} \pmod 4$.

5. Let the number of glasses purchased be n. Then, it cost Sam \$$5n$ to buy the glasses. Given that he made a 20% profit, that means he sold the $n - 5$ unbroken glasses for a total of \$$6n$ at \$8 each. Therefore, we have $8(n - 5) = 6n$, or $n = \boxed{20}$.

6. Since $R = \{3, 5\}$ and $R \cup T = \{1, 3, 4, 5\}$, T must contain at least $\{1, 4\}$ and cannot contain 2 or 6. Additionally, $S \cap T = \{3\}$, so T must also contain $\{3\}$. Finally, we are given that $S \cup T = \{1, 2, 3, 4, 6\}$, so T also cannot contain 5. Therefore, we know that $T = \{1, 3, 4\}$.

   Using the fact that $S \cap T = \{3\}$, we know that S contains 3 but not $\{1, 4\}$. Since $S \cup T = \{1, 2, 3, 4, 6\}$, we also know that S must contain 2 and 6 but not 5. Therefore, $S = \boxed{\{2, 3, 6\}}$.

7. At 300 miles per 10 gallons, Henry gets 30 miles per gallon. Therefore, at 55 mph, he gets $30 + 3 = 33$ mpg. This is equal to $\frac{33}{55} = \frac{3}{5}$ hours per gallon. At this rate, it will take him $25 \cdot \frac{3}{5} = \boxed{15}$ hours to burn 25 gallons of gasoline.

8. We know that 22! is a multiple of both 9 and 11, so we can use the divisibility rules. The divisibility rule for 9 is that the sum of the digits is divisible by 9 (stemming from the fact that $10^n \equiv 1 \pmod 9$ for $n \in \mathbb{N}$) and the divisibility rule for 11 is that the sum and difference of alternating digits is divisible by 11 (since $10^n \equiv (-1)^n \pmod{11}$ for $n \in \mathbb{N}$).

   For 9, we have that $a+b = 6$ or 15. For 11, we have that $a-b = 6$ or -5. Remembering that a and b are integers from 0 through 9, the only possibility that makes sense is that $a + b = 6$ and $a - b = 6$. Therefore, $(a, b) = \boxed{(6, 0)}$.